NONABELIAN CHABAUTY

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1. Multiple polylogs

Define

$$\mathcal{L}_{(k_1,\dots,k_m)}(z) := \sum_{0<n_1<\dots<n_m} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}.$$

For example,

$$\mathcal{L}_{(k)}(z) = \sum_{n>0} \frac{z^n}{n^k}.$$

Their special values are related to zeta and L-values. But we will be interested in the functions themselves.

Index them with words w on $\{A, B\}$. Define

$$\begin{aligned}
\mathcal{L}_{\emptyset}(z) &= 1\\
\mathcal{L}_{A^n}(z) &= \frac{1}{n!}(\log z)^n\\
\mathcal{L}_{Aw}(z) &= \int_0^z \frac{dt}{t}\mathcal{L}_w(t), \quad \text{if } w \neq A^n\\
\mathcal{L}_{Bw}(z) &= \int_0^z \frac{dt}{1-t}\mathcal{L}_w(t), \quad \text{if } w \neq A^n.
\end{aligned}$$

These are multivalued functions on $\mathbb{P}^1 - \{0, 1, \infty\}$. For z near 0, we have $\mathcal{L}_{(k_1,\dots,k_m)} = \mathcal{L}_w$ where $w := A^{k_1-1}B\cdots A^{k_m-1}B$.

Define

$$G(z) = \sum_w \mathcal{L}_w(z)[w],$$

a function with values in $\mathbb{C}\langle\langle A, B\rangle\rangle$. Then $dG = \left(A\frac{dz}{z} + B\frac{dz}{1-z}\right)G(z)$. One can define p-adic versions. The coefficients are then p-adic multiple polylogs.

2. S-unit equations

Let S be a finite set of primes. Consider solutions to $x + y = 1$ with $x, y \in \mathbb{Z}[1/S]^\times$.

If $S = \{\infty, \ell, q\}$ and $p \notin S$, There exists a polynomial $P(\mathcal{L}_w)$ in the $\mathcal{L}_w$ with $|w| \leq 4$ and having $\mathbb{Q}_p$-coefficients such that $P(\mathcal{L}_w)(x) = 0$ for every solution $(x, y) = 0$.

For larger S, we need $|w| \leq N$, where N is explicitly computable.

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3. Integral points on an elliptic curve

Let $\mathcal{E}$ be the affine curve $y^2 = x^3 + 2$ over $\mathbb{Z}$. Let $\alpha = \frac{dx}{y}$ and $\beta = \frac{x\,dx}{y}$. Let $p = 5$. The fundamental group of $\mathcal{E}(\mathbb{C})$ is a free group on two generators. Let $a = (-1, 1)$. Define Coleman functions

$$\mathcal{L}_A(z) = \int_a^z \frac{dx}{y}$$
$$\mathcal{L}_B(z) = \int_a^z \frac{x\,dx}{y}$$
$$\mathcal{L}_{AB}(z) = \int_a^z \frac{dx}{y}\mathcal{L}_B$$

on $\mathcal{E}(\mathbb{Z}_p)$. Then there exists a polynomial $P(\mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_{AB})$ such that $P(\mathcal{L})(x, y) = 0$ for $(x, y) \in \mathcal{E}(\mathbb{Z})$.

4. General curves

Let $\mathbb{R} = \mathbb{Z}[1/S]$. Let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth proper curve of genus g minus a finite étale divisor whose fibers are of degree t. So t is "the number of points at infinity". Define

$$m = \begin{cases} 2g & \text{if } t = 0 \\ 2g + t - 1 & \text{if } t > 0 \end{cases}$$

Pick $p \notin S$. One can define $\mathcal{L}_w(z)$ where w runs over words on $\{A_1, \ldots, A_m\}$ on $\mathcal{X}(\mathbb{Z}_p)$. The A_i correspond to generators α_i of $H^1_{\mathrm{dR}}(\mathcal{X})$.

Assume "motivic conjectures" (e.g., Bloch-Kato on surjectivity on p-adic Chern class maps or Fontaine-Mazur conjecture on representations of geometric origin). Then we can compute $N = N(\mathcal{X}, S, p)$ such that there exists a polynomial $P(\mathcal{L}_w)$ in the $\mathcal{L}_w$ for $|w| \leq N$ such that $P(\mathcal{L}_w)(x) = 0$ for all $x \in \mathcal{X}(R)$. This would imply the theorems of Faltings and Siegel for curves over $\mathbb{Q}$.

5. Origin of polylogs

They come from algebraic functions on classifying spaces associated to unipotent π_1's. Let X be a variety over a number field. Fix $b, x \in X$. Let $U^M = \pi_1^M(\overline{X}, b)$ and $P^M(x) = \pi_1^M(\overline{X}; b, x)$.

For a topological space X and points b, x, the space $\pi_1(X; b, x)$ is a torsor for $\pi_1(X; b)$. We have a map from X to a classifying space of torsors sending x to $\pi_1(X; b, x)$. Analogous statements hold for the other manifestations of π_1.

As in Kiran's talk we have

$$\begin{array}{ccccc}
\mathcal{X}(R) & \longrightarrow & \mathcal{X}(R_p) & \xrightarrow{\text{multiple polylogs}} & \\
\downarrow & & \downarrow & & \\
\mathcal{C}^{\mathrm{et}}_{\mathrm{glob}} & \longrightarrow & \mathcal{C}^{\mathrm{et}}_{\mathrm{loc}} & \overset{p\text{-adic Hodge theory}}{=\!=\!=\!=} & \mathcal{C}^{\mathrm{dR}}
\end{array}$$

where the space at lower left is nonabelian cohomology. The left map sends x to $\pi_1^{\mathrm{et,un}}(\overline{X}; b, x)$.

The objects in the bottom row map to corresponding quotients for each n (we add a subscript n to each object), and these quotients are algebraic varieties. The vertical (and diagonal) maps are transcendental. The diagonal map has Zariski-dense image.

Inside $\mathcal{C}_n^{\mathrm{dR}}$ the intersection of the images of $\mathcal{C}_{\mathrm{glob}}^{\mathrm{et}}$ and $\mathcal{X}(R_p)$ should be finite. This is true for $\mathbb{P}^1 - \{0,1,\infty\}$ and for elliptic curves of rank 1, and in general assuming motivic conjectures.

There exists a nonzero algebraic function α on $\mathcal{C}_n^{\mathrm{dR}}$ such that α restricted to the image of $\mathcal{C}_{\mathrm{glob}}^{\mathrm{et}}$ is 0. But $\mathcal{X}(R_p)$ has Zariski-dense image in $\mathcal{C}_n^{\mathrm{dR}}$, so α pulls back to a nonzero function on $\mathcal{X}(R_p)$, and hence has finitely many zeros. Thus we get a bound on $\mathcal{X}(R)$.

6. Profinite case

Let $b \in Z$ be a variety. Let $\mathrm{Cov}(Z)$ be the set of finite étale covers Y of Z. Let F_b be the functor sending $Y \to Z$ to the fiber Y_b considered as a finite set. Define $\hat{\pi}_1(Z,b) := \mathrm{Aut}(F_b)$. Define $\hat{\pi}_1(Z;b,x) := \mathrm{Isom}(F_b,F_x)$; this is a torsor for $\hat{\pi}_1(Z,b)$ with continuous action.

If $Z = \overline{X}$ where X is over $\mathbb{Q}$ and $x, b \in X(\mathbb{Q})$, then $G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{Cov}(\overline{X})$, so G acts on $\hat{\pi}_1(Z,b)$ and $\hat{\pi}_1(Z;b,x)$. In particular, the latter is a G-equivariant torsor for $\hat{\pi}_1(Z,b)$; such G-equivariant torsors T are classified by $H^1(G,\hat{\pi}_1(\overline{X},b))$. Namely, given T, choose $t \in T$ and for each $g \in G$, find the $\gamma_g \in \hat{\pi}_1(\overline{X},b)$ such that $g(t) = t\gamma_g$; then $g \mapsto \gamma_g$ is a 1-cocycle representing an element of $H^1(G,\hat{\pi}_1(\overline{X},b))$.

Let Z be a variety. Let $\tilde{Z}$ be its universal covering (as a pro-variety, represented by a cofinal inverse system of Z_i's).

For $x \in Z$,

$$\tilde{Z}_x \simeq \hat{\pi}_1(Z;b,x).$$

If $Y \to Z$ and $y \in Y_b$, then there exists $\phi_y \colon \tilde{Z} \to Y$ mapping $\tilde{b}$ to y, and $\phi_y(\tilde{x}) \in Y_x$.

For an arbitrary manifold M,

$$M \leftarrow \tilde{M} = \bigcup_{m \in M} \pi_1(M;b,x).$$

Let $X/\mathbb{Q}$ be a variety and $b \in X(\mathbb{Q})$. Then we have a map

$$\begin{aligned} X(\mathbb{Q}) &\to H^1(G,\hat{\pi}_1(\overline{X},b)) \\ x &\mapsto [\hat{\pi}_1(\overline{X};b,x)]. \end{aligned}$$

If $X = Z$ is an elliptic curve, then $\hat{\pi}_1(\overline{E},e)$ is the $\hat{\mathbb{Z}}$ Tate module $T(\overline{E})$. The map $E(\mathbb{Q}) \to H^1(G,\hat{\pi}_1(\overline{E},e))$ is the map from Kummer theory. (The H^1 is defined as an inverse limit, or using continuous cocycles.) Conjecturally it is an isomorphism (this is equivalent to the finiteness of the p-primary part of $\text{Ш}(E)$ for all p).

If $X/\mathbb{Q}$ is a curve of genus ≥ 2, then

$$X(\mathbb{Q}) \to H^1(G,\hat{\pi}_1(\overline{X},b))$$

is conjectured to be a bijection. (This is the version of Grothendieck's section conjecture in which a base point is fixed. This can be viewed as a nonabelian analogue of the conjecture that Ш is finite.) It is injective by the Mordell-Weil theorem for the Jacobian of X.

Remark 6.1. For any variety V over $\mathbb{Q}$ with $b \in V(\mathbb{Q})$, injectivity of $V(\mathbb{Q}) \to H^1(G,\hat{\pi}_1(\overline{V},b))$ should be viewed as a "nonabelian Mordell-Weil theorem".

7. Unipotent version

Let $X/\mathbb{Q}$ be a smooth curve.

Let $U = U^{\text{et}} = U^{\text{et}}(\overline{X})$ be the pro-unipotent completion of $\hat{\pi}_1(\overline{X}, b)$ over $\mathbb{Q}_p$. The groups $U_n = L^{n+1}\backslash U$ are algebraic groups over $\mathbb{Q}_p$. Let E be the completed universal enveloping algebra of $\operatorname{Lie} U^{\text{et}}$. Let $I \subseteq E$ be the augmentation ideal (the ideal generated by the Lie algebra). Let $E_n = E/I^{n+1}$. Then E is a projective system of continuous $\mathbb{Q}_p$-representation of $\hat{\pi}_1(\overline{X}, b)$. We get an étale pro-sheaf $\mathcal{E}$ on $\overline{X}$ such that $\mathcal{E}_b \simeq E$. Let $e \in E$ be the identity of the group. The pair $(\mathcal{E}, e)$ is the universal $\mathbb{Q}_p$ unipotent locally constant sheaf on $\overline{X}$: for any other pair $(\mathcal{L}, \ell)$ with $\ell \in \mathcal{L}_b$, there is a unique $\mathcal{E} \to \mathcal{L}$ mapping e to ℓ. In particular, there is a unique map $\mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ sending e to $e \otimes e \in (\mathcal{E} \otimes \mathcal{E})_b$.

Let $U_n(\overline{X})$ be the set of locally constant sheaves $\mathcal{L}$ of $\mathbb{Q}_p$-vector spaces that are *unipotent*, i.e., admitting a filtration $\mathcal{L} = \mathcal{L}^0 \supset \mathcal{L}^1 \supset \cdots \supset \mathcal{L}^{n+1} = 0$ with $\mathcal{L}^i/\mathcal{L}^{i+1} \simeq \mathbb{Q}_p^{r_i}$ (a trivial local system). Define

$$P^{\text{et}}(x) = \pi_1^{\text{et},u}(\overline{X}, b, x) := \operatorname{Isom}(F_b, F_x)$$

where F_x is the functor from $U_n(\overline{X})$ to $\mathbb{Q}_p$-vector spaces taking $\mathcal{L}$ to $\mathcal{L}_x$. Then $\pi_1^{\text{et},u}(\overline{X}; b, x)$ is the set of grouplike elements in $\mathcal{E}_x$.

Given $b, x \in X(\mathbb{Q})$, the set $P_n^{\text{et}}(x)$ carries a G-action, and $[P_n^{\text{et}}(x)] \in H^1(G, U_n^{\text{et}})$. We get $X(\mathbb{Q}) \to H^1(G, U_n^{\text{et}})$.

Given $\mathcal{X} \to R = \mathbb{Z}[1/S]$ and $p \notin S$, $T := S \cup \{p\}$, we get

$$[P_n^{\text{et}}(x)] \in H^1(G_T, U_n^{\text{et}})$$

where $G_T := \operatorname{Gal}(\mathbb{Q}_T/\mathbb{Q})$ where $\mathbb{Q}_T$ is the maximal extension unramified outside T.

Let $U_n = U/U^n$. We have

$$0 \to \frac{U^n}{U^{n+1}} \to U_{n+1} \to U_n \to 0$$

and $U^1 = U$, $U^2 = [U, U]$. So $U_2 = U/[U, U] \simeq H^1(\overline{X}, \mathbb{Q}_p) = H^1(\overline{X}, \mathbb{Q}_p)^\wedge$. We have

$$0 \to H^1(G_T, U^n/U^{n+1}) \to H^1(G_T, U_{n+1}) \to H^1(G_T, U_n) \to H^2(G_T, U^n/U^{n+1}).$$

In particular,

$$0 \to H^1(G_T, U^2/U^3) \to H^1(G_T, U_3) \to H^1(G_T, U_2) \xrightarrow{\delta} H^2(G_T, U^2/U^3).$$

Here δ is an algebraic map between $\mathbb{Q}_p$-varieties.

Consider $X(R) \hookrightarrow X(R_p)$. Effectively separate the p-adic distance between points in $X(R)$. This would lead to an effective injection $X(R) \to J(R)/NJ(R)$.

$$\begin{array}{ccc} & \nearrow & H^1(\pi) \\ & & \downarrow \\ X(R) & \longrightarrow & H^1(\pi^{\text{ab}}) \\ & & \uparrow \\ & \searrow & J(R) \end{array}$$

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_p) \\ \downarrow & & \downarrow \\ H^1(G_T, U_n^{\mathrm{et}}) & \longrightarrow & H^1(G_p, U_n^{\mathrm{et}}) \end{array}$$

$$0 \to H^1(G_T, U^n/U^{n+1}) \to H^1(G_T, U_{n+1}) \to H^1(G_T, U_n) \xrightarrow{\delta} H^2(G_T, U^n/U^{n+1})$$

One can define variety structures on these such that $H^1(G_T, U_{n+1}) \simeq Z_n \times H^1(G_T, U^n/U^{n+1})$ where $Z_n = \delta^{-1}(0)$.

For $\mathbb{P}^1 - \{0, 1, \infty\}$, it turns out (by a deep theorem of Soulé) that $\delta = 0$. Then $H^1(G_T, U_n) \simeq H^1(G_T, U_2) \times H^1(G_T, U^2/U^3) \times \cdots \times H^1(G_T, U^{n-1}/U^n)$, and this maps to U_n^{dR}/F^0.

We can make

$$H^1(G_T, U_n) \xrightarrow{\delta} H^2(G_T, U^n/U^{n+1})$$

from

$$0 \to U^n/U^{n+1} \to U_{n+1} \to U_n \to 0.$$

Let ω be a 2-cocycle for U_n with values in U^n/U^{n+1} representing this extension. Then $\delta(c)(g_1, g_2) = \omega(c(g_1), c(g_2)) g_1^{-1} \alpha(g_1, c(g_2))$, where $\alpha \colon U_n \times G_T \to U^n/U^{n+1}$ is defined by $\alpha(g, u) = g(\tilde{u})\tilde{u}^{-1}$, where $u \mapsto \tilde{u}$ is a splitting $U_n \to U_{n+1}$ of the surjection above.

8. de Rham picture

Let F be a field of characteristic 0. Let X/F be a smooth affine curve. Let $\alpha_1, \ldots, \alpha_m$ be regular 1-forms giving a basis of $H^1_{\mathrm{dR}}(X)$. Then $U_n^{\mathrm{dR}}(X)$ is the category of unipotent vector bundles with flat connection $(\mathcal{U}, \nabla)$; i.e.,

$$\mathcal{U} = \mathcal{U}^0 \supset \mathcal{U}^1 \supset \cdots \supset \mathcal{U}^{n+1} = 0$$

with $(\mathcal{U}^i/\mathcal{U}^{i+1}, \nabla) \simeq (\mathcal{O}_X^r, d)$.

Fact: $(\mathcal{U}, \nabla) \simeq (\mathcal{O}_X^r, d + \sum_{i=1}^m N_i \alpha_i)$ where the N_i are constant strictly upper triangular matrices.

Fix $b \in X(F)$. Let $F_b \colon U_n^{\mathrm{dR}}(X) \to$ (F-vector spaces) be the fiber functor sending $(\mathcal{U}, \nabla)$ to $\mathcal{U}_b$.

Let $F\langle\langle A \rangle\rangle$ be the algebra of free noncommutative power series in variables A_i. There is a comultiplication $F\langle\langle A \rangle\rangle \to F\langle\langle A \rangle\rangle \hat{\otimes} F\langle\langle A \rangle\rangle$ sending A_i to $A_i \otimes 1 + 1 \otimes A_i$. Let $U^{\mathrm{dR}}(X) = \pi^{\mathrm{dR}}(X; b) := \mathrm{Aut}^{\otimes}(F_b)$. Fact: $U^{\mathrm{dR}}(X)$ is isomorphic to the set of grouplike elements in $F\langle\langle A \rangle\rangle := F\langle\langle A_1, \ldots, A_m \rangle\rangle$

Also define $P^{\mathrm{dR}}(x) = \pi_1^{\mathrm{dR}}(X; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x)$. This is isomorphic to the set of grouplike elements in $F\langle\langle A \rangle\rangle$.

If $F = \mathbb{C}$, then

$$[\gamma] \in P^{\mathrm{dR}}(x) \subset \mathbb{C}\langle\langle A \rangle\rangle$$

Parallel transport along γ is $P(\gamma) \colon V_b \simeq V_x$.

$$[\gamma] = \sum \int_\gamma \alpha_w [w].$$

Here $w = A_{i_1} \cdots A_{i_k}$ and $\int_\gamma \alpha_w = \int_\gamma \alpha_{i_1} \cdots \alpha_{i_k}$.

Define the discrete subgroup of topological paths $L^{\mathrm{dR}} \subset U_n^{\mathrm{dR}}(X) \supset F^0 U^{\mathrm{dR}}$.

Consider triples (T, L_T, F^0) where T is a torsor for U^{dR}, F^0 is given by the Hodge filtration on T and is an F^0U^{dR}-torsor, and $L_T \subset T$ is an L^{dR}-torsor.

Such triples are classified by $F^0\backslash U_n^{\mathrm{dR}}(x)/L$. These are the higher Albanese varieties that Hain introduced. There is a map $\theta\colon X(\mathbb{C}) \to F^0\backslash U_n^{\mathrm{dR}}(x)/L$ sending x to $[P^{\mathrm{dR}}(x)]$.

In the case of $\mathbb{P}^1 - \{0, 1, \infty\}$ (containing b and x), we have $U^{\mathrm{dR}} \subset \mathbb{C}\langle\langle A, B\rangle\rangle$ and F^0 turns out to be 0. Then $\theta(x) = [\sum \int_\gamma \alpha_w[w]] \in U^{\mathrm{dR}}/L$.

Now suppose $X/\mathbb{Q}_p$. We have $U^{\mathrm{dR}} \supset F^0$ with ϕ-action. Consider $P^{\mathrm{dR}}(x) \simeq P^{\mathrm{dR}}(x \bmod p)$. Consider (T, F^0, ϕ) where ϕ acts compatibly on U and T. Then we have

$$F^0\backslash U^{\mathrm{dR}}/(U^{\mathrm{dR}})^{\phi=\mathrm{id}}$$

with $(U^{\mathrm{dR}})^{\phi=\mathrm{id}} = \{e\}$.

We get

$$\theta\colon X(\mathbb{Z}_p) \to F^0\backslash U^{\mathrm{dR}}$$

defined $\theta(x) = [\sum_w \int_b^x \alpha_w[w]]$.

$$\begin{array}{ccccc}
X(R) & \longrightarrow & X(R_p) & & \\
\downarrow & & \downarrow & \searrow & \\
H^1(G_T, U_n^{\mathrm{et}}) & \longrightarrow & H^1_f(G_p, U_n^{\mathrm{et}}) & \xrightarrow{D} & F^0\backslash U_n^{\mathrm{dR}}
\end{array}$$

$H^1_f(G_p, U_n^{\mathrm{et}})$ is the set of elements of $H^1(G_p, U_n^{\mathrm{et}})$ representing torsors that trivialize over B_{crys}.

And $D(T) := \operatorname{Spec}((\mathcal{T} \otimes B_{\mathrm{crys}})^{G_p})$ where $T = \operatorname{Spec} \mathcal{T}$.

9. Example

Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$. Let $T = S \cup \{p\}$.

$$\begin{array}{ccccc}
H^1(G_T, U_3) & \longrightarrow & H^1_f(G_p, U_3) & \longrightarrow & U_3^{\mathrm{dR}} \\
\| & & & & \downarrow \\
H^1_f(G_T, U_2) & \longrightarrow & H^1_f(G_p, U_2) & = & H_1^{\mathrm{dR}}(X)
\end{array}$$

The group U_2 is $\mathbb{Q}_p(1)^2$, and $H^1(G_T, U_2) = \mathbb{Z}[T^{-1}]^\times \times \mathbb{Z}[T^{-1}]^\times$ and $H^1_f(G_T, U_2) = \mathbb{Z}[S^{-1}]^\times \times \mathbb{Z}[S^{-1}]^\times$ and $H^1_f(G_p, U_2) = \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. The vertical isomorphism at left is there because $H^1(G_F, U^2/U^3) \simeq H^1(G_T, \mathbb{Q}_p(2)) = 0$.

Recall $U_n = U/U^n$. We have

$$0 \to \frac{U^n}{U^{n+1}} \to U_{n+1} \to U_n \to 0.$$

Let $r_n = \dim \frac{U^n}{U^{n+1}}$.

For a genus-g curve minus t points (with $t > 0$), set $m = 2g + t - 1$. Then $\sum_{i \mid n} i r_i = m^n$. For example, for $\mathbb{P}^1 - \{0, 1, \infty\}$, we have $m = 2$, and $r_1 = 2$, $r_1 + 2r_2 = 4$ so $r_2 = 1$, $r_1 + 3r_3 = 8$ so $r_3 = 2$, $r_1 + 2r_2 + 4r_4 = 16$ so $r_4 = 3$.

$$\dim H^1(G_T, U^n/U^{n+1}) - \dim H^2(G_T, U^n/U^{n+1}) = \dim(U^n/U^{n+1})^-$$

where the $-$ means the minus part for complex conjugation. We have $U^n/U^{n+1} \simeq \mathbb{Q}_p(n)^{r_n}$, and $H^2(G_T, \mathbb{Q}_p(n)) = 0$ for all $n \geq 2$, so $\dim H^1(G_T, U/U^2) = 2(|T|-1)$, and $\dim H^1(G_T, U^n/U^{n+1})$ is r_n if n is odd and 0 if n is even. Thus

$$\dim H^1(G_T, U_n) = 2(|T|-1) + r_3 + r_5 + \cdots$$
$$\dim U_n = r_1 + r_2 + \cdots + r_{n-1},$$

so $\dim U_n > \dim H^1(G_T, U_n)$ eventually.

10. Elliptic curve analogue

Let E be an elliptic curve over $\mathbb{Q}$ of rank 1. Let $X = E - \{e\}$, e.g., $y^2 = x^3 + 2$.

$$\begin{array}{ccccc} H^1_f(G_T, U_2/U_3) & \longrightarrow & U_3^{\mathrm{dR}}/F_0 & = & \mathbb{Q}_p(1) \\ \downarrow & & \downarrow & & \\ H^1_f(G_T, U_3) & \longrightarrow & U_3^{\mathrm{dR}}/F_0 & & \\ \downarrow & & \downarrow & & \\ H^1_f(G_T, U_2) & \longrightarrow & U_2^{\mathrm{dR}}/F_0 & & \end{array}$$

The space at lower right is 1-dimensional, so the space above it is 2-dimensional. The closure of the image of $X(\mathbb{Z})$ maps into $H^1_\Sigma(G_T, U_3)$ and $H^1_\Sigma(G_T, U_2)$.

When $n = 2$, the image of $X(\mathbb{Z}_\ell) \to H^1(G_\ell, U_n)$ (where U_n is unipotent over $\mathbb{Q}_p$) is finite (Tamagawa). This forces the global image to satisfy additional conditions.