Enumerating alternating trees

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ENUMERATING ALTERNATING TREES

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Abstract. In this paper we examine the enumeration of alternating trees. We give a bijective proof of the fact that the number of alternating unrooted trees with n vertices is given by $\frac{1}{n2^{n-1}}\sum_{k=1}^{n}\binom{n}{k}k^{n-1}$, a problem first posed by Postnikov in [4]. We also show that the number of alternating ordered trees with n vertices is $2(n-1)^{n-1}$.

1. Introduction and main results

Definition 1.1. *A tree T on the set of vertices $[n]=\{1,2,\cdots,n\}$ is said to be alternating if for every path $x_1,x_2,x_3,x_4,\cdots$ in T we have $x_1<x_2>x_3<x_4>\cdots$ or $x_1>x_2<x_3>x_4<\cdots$.*

In this paper, unless specified, we consider trees having a distinguished vertex, called the *root*.

In a recent work [4], Postnikov gives a formula for the number of unrooted alternating trees (called intransitive trees in his paper) and a functional equation satisfied by their generating function. These results are summarized in the next two theorems.

Theorem 1.1. [4] *Let F_n be the number of unrooted alternating trees on $[n]$. Then*

$$F_n=\frac{1}{n2^{n-1}}\sum_{k=1}^{n}\binom{n}{k}k^{n-1}.$$

It is clear from the definition of rooted and unrooted trees that the number of (rooted) alternating trees on $[n]$ is nF_n. Now let $F(t)$ be the shifted generating function of unrooted alternating trees:

$$F(t)=\sum_{n\geq 0}F_{n+1}\frac{t^n}{n!}.$$

Theorem 1.2. [4] *$F(t)$ satisfies the following functional equation:*

$$F=e^{\frac{t}{2}(F+1)}.$$

To prove Theorem 1.1, Postnikov first proves Theorem 1.2 and then solves the equation using the Lagrange Inversion Formula. In Section 2 we present a bijective proof of Theorem 1.1, which answers a problem posed by Postnikov.

In Section 3, we consider *ordered trees*: a (rooted) ordered tree is a (rooted) tree such that the sons of each vertex are linearly ordered and an unrooted ordered tree is an unrooted tree such that, for each vertex x, the neighbors of x (the vertices linked to x with an edge) are cyclically ordered. We prove the following results.

Theorem 1.3. *Let G_n be the number of alternating ordered trees on $[n]$. Then*

$$G_n=2(n-1)^{n-1}.$$

Corollary 1.1. *Let H_n be the number of unrooted alternating ordered trees on $[n]$. Then*

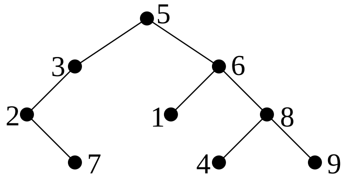
$$H_n=(n-1)^{n-2}.$$

2. A bijective proof of Theorem 1.1

In order to give a bijective proof of Theorem 1.1, we need to introduce the local binary search (LBS) trees, used by Postnikov in [4].

Definition 2.1. *A LBS tree is a binary ordered tree such that every left son has a smaller label than its parent, and every right son has a larger label than its parent.*

For example, the following tree is a LBS tree with 9 vertices.



Theorem 2.1. [4] *The number of LBS trees on* $[n]$ *such that the root has only one son is*

$$nF_n = \frac{1}{2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}.$$

To prove this result, Postnikov gives a bijection, called ϕ, between LBS trees on $[n]$ such that the root has only one son and alternating trees on $[n]$ (we recall that there are nF_n such trees) and uses Theorem 1.1.

Now, we give a bijective proof of Theorem 2.1, which, combined with the bijection ϕ of Postnikov, provides a complete bijective proof of Theorem 1.1. In fact we describe a bijection Φ between two families of trees, $\mathcal{B}_n$ and $\mathcal{T}_n$, that we define below.

We denote by $\mathcal{B}_n$ the set of *bicolored LBS trees* on $[n]$ such that the root has only one son: every non-root vertex can be white colored or black colored, the root being a black vertex. It follows immediately from the bijection ϕ that

$$|\mathcal{B}_n| = 2^{n-1} nF_n. \tag{1}$$

We denote by $\mathcal{T}_n$ the set of bicolored trees on $[n]$ such that every internal vertex (vertices having at least one son) is black and every leaf (vertices with no son) can be white or black. For a tree T of $\mathcal{T}_n$, we denote by $int(T)$, $leaves_B(T)$ and $leaves_W(T)$ respectively its number of internal vertices, its number of black leaves and its number of white leaves. From the Prüfer encoding of trees (see [5] or [3]), we can say that the trees of $\mathcal{T}_n$ such that $int(T) + leaves_B(T) = k$ $(1 \le k \le n)$ are in bijection with the set of pairs (A, w) where A is a subset of $[n]$ such that $|A| = k$ and w is a word of length $n-1$ on the alphabet A. Therefore, the number of trees of $\mathcal{T}_n$ such that $int(T) + leaves_B(T) = k$ is

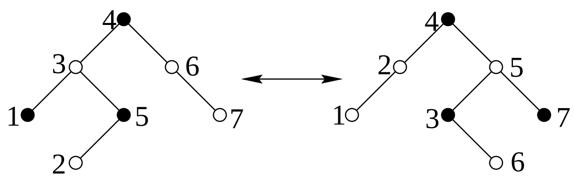
$$\binom{n}{k} k^{n-1},$$

and

$$|\mathcal{T}_n| = \sum_{k=1}^{n} \binom{n}{k} k^{n-1}. \tag{2}$$

The first step in the description of the bijection Φ is an involution Θ on LBS trees of $\mathcal{B}_n$. Let T be a LBS tree on the set of vertices $\{x_1, x_2, \cdots, x_k\}$, with $x_1 < x_2 < \cdots < x_k$. The LBS tree $\Theta(T)$ is obtained by performing the following operations:

- swap labels x_i and x_{k-i+1} for $1 \le i \le k$,
- for every vertex, swap its right subtree and its left subtree,
- the colors on the vertices stay attached to the vertices and not to the labels.



We can now give a recursive description of the bijection Φ between $\mathcal{T}_n$ and $\mathcal{B}_n$. We suppose that the input is a tree T of $\mathcal{T}_n$ and we want as output a tree $\Phi(T) = B$ of $\mathcal{B}_n$.

1. If T has only one vertex, then $B = T$ (the color of the vertex doesn't change).

2. If T has two vertices, then we distinguish two cases:

- if T is the tree with root labeled 1 and a leaf labeled 2, then B is the bicolored LBS tree having root labeled 1 with a right son labeled 2, the colors of the two vertices being unchanged,

- otherwise, T has root labeled 2 and a leaf labeled 1, B is the bicolored LBS tree having root labeled 2 with a left son labeled 1 the colors of the two vertices being unchanged.

Remark 2.1. From these first two cases, we can use the hypothesis that the root of $\Phi(T)$ has the same label than the root of T and has at most one son.

3. In the general case, T has at least 3 vertices. Let r be the root, $x_1, x_2, \cdots, x_k$ its sons and $T_1, T_2, \cdots, T_k$ the subtrees of T with respective roots $x_1, x_2, \cdots, x_k$. For each tree T_i, we denote its image $\Phi(T_i) = B_i$ (using Remark 2.1, we can suppose that the root of B_i is x_i and has at most one son) and we define the tree B'_i by the following rules:

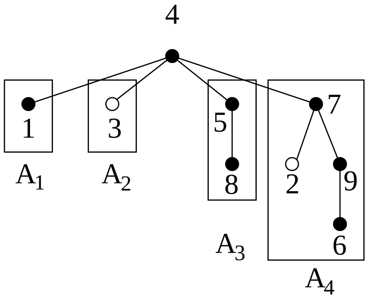
- if B_i has only one vertex x_i, then $B'_i = B_i$,
- if x_i has only one right son, then B'_i is B_i the root being black colored,
- if x_i has only one left son, then $B'_i = \Theta(B_i)$, the root being black colored (the root of B'_i can differ from the root of T and has only a right son).

Now we have a set of k bicolored LBS trees such that the root has no left son. We denote by $y_1, y_2, \cdots, y_k$ their roots such that $y_1 < y_2 < \cdots < y_k$, and we end the construction of B:

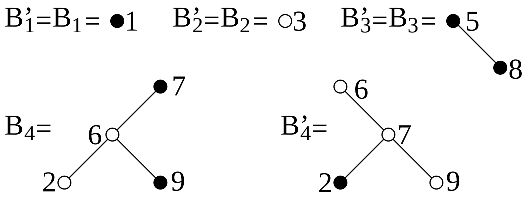
- if $y_k < r$, set y_k to be the left son of r, y_{k-1} the left son of y_k, etc,
- if $y_k > r$, set y_k to be the right son of r, y_{k-1} the left son of y_k, y_{k-2} the left son of y_{k-1}, etc.

Remark 2.2. It is immediate to check the validity of the hypothesis about the root of $\Phi(T)$ given in Remark 2.1, and the fact that the root of $\Phi(T)$ is black.

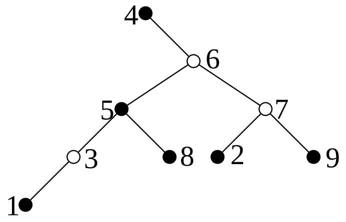
Now, we give an example. Let T be the following tree of $\mathcal{T}_{10}$.



Then we have



and finally, $\Phi(T)$ is the following tree.



The construction of the inverse map is clear and we have a bijective proof of Theorem 1.1.

To conclude this section, we give some additional enumerative results on the trees considered above.

Corollary 2.1. *Let $\mathcal{B}_{n,p}$ be the set of LBS trees on $[n]$ such that the root has only one son x and the left branch from x (the path issued from u and following only left edges) has p vertices.*

$$|\mathcal{B}_{n,p}| = \frac{n}{2^{n-1}} \binom{n-2}{p-1} \sum_{k=0}^{n-1} \binom{n-1}{k} k^{n-1-p}.$$

Proof. From the definition of Φ, we can say that $2^{n-1}|\mathcal{B}_{n,p}|$ is the number of bicolored trees on $[n]$ such that the root has exactly p sons, every internal vertex is black colored and the leaves are white or black colored. The result follows immediately from the Prüfer encoding of such trees and an easy computation. □

Corollary 2.2. *The number of trees on $[n]$ such that every vertex has at least one son greater than it (in other words, for every vertex u there is an increasing path from u to a leaf) is*

$$\frac{1}{2}nF_n = \frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}k^{n-1}.$$

Proof. This result is a consequence of the limitation of Φ on LBS trees such that the root has only a right son and every vertex is black colored. In this way, we recognize the classical correspondence between binary trees and trees (see [2]) limited to LBS trees. □

3. Enumerating alternating ordered trees

In order to prove Theorem 1.3, we will focus on the family of alternating ordered trees such that the root is lower than its sons. Let L_n denote the number of such trees on $[n]$. We will show that $L_n = (n-1)^{n-1}$, and then deduce immediately Theorem 1.3. We denote by $L(t)$ the exponential generating function of such trees:

$$L(t) = \sum_{n\geq 1} L_n \frac{t^n}{n!}.$$

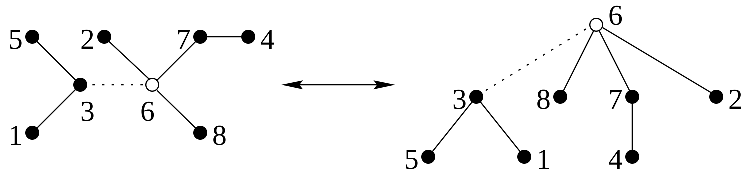
Lemma 3.1. *The generating function L satisfies the functional equation*

$$L(t) - 1 = -e^{t/(L(t)-1)}.$$

Proof. We have clearly the following relation between G_n and L_n:

$$G_1 = L_1 = 1,\ G_n = 2L_n\ (n>1). \tag{3}$$

Now, remark that there is a bijection from the set of triples $(T,(x,y),z)$ where T is an unrooted ordered tree, (x,y) an edge of T and z a vertex of this edge (either x or y) to the set of (rooted) ordered trees: we can associate an ordered tree to such a triple by rooting T in the vertex z, with the condition that the other vertex of the edge (x,y) is the leftmost son of z, as shown in the following figure (where the edge (x,y) is the dotted edge $(3,6)$ and $z=6$).



Then we have the following relation between G_n and H_n:

$$G_1 = H_1 = 1,\ G_n = 2(n-1)H_n\ (n>1), \tag{4}$$

and between L_n and H_n (consequence of (3) and (4)):

$$L_1 = H_1 = 1,\ L_n = (n-1)H_n\ (n>1). \tag{5}$$

Finally, we introduce the shifted exponential generating function of unrooted alternating ordered trees $H(t)$

$$H(t) = \sum_{n\geq 0} H_{n+1}\frac{t^n}{n!}.$$

Using classical results on exponential generating functions and labeled structures (see [1]), we can say that:

$$H(t) = 1 + \ln\left(\frac{1}{1-L(t)}\right),$$

and then

$$\frac{\partial H(t)}{\partial t} = \frac{1}{1-L(t)}\frac{\partial L(t)}{\partial t}. \tag{6}$$

The rest of the proof is calculus using the previous relations. From (5) we have

$$\sum_{n\geq 1} L_n \frac{t^n}{n!} = t + \sum_{n\geq 2}(n-1)H_n\frac{t^n}{n!}$$

which implies that

$$L(t) = tH(t) - \sum_{n\geq 2} H_n \frac{t^n}{n!}.$$

By differentiation and using equation (6), we obtain that:

$$\frac{\partial L(t)}{\partial t} = \frac{1}{L(t)-1}\left(L(t)-1-t\frac{\partial L(t)}{\partial t}\right)$$

$$\Longrightarrow \frac{1}{(L(t)-1)^2}\left(L(t)-1-t\frac{\partial L(t)}{\partial t}\right) = \frac{1}{(L(t)-1)}\frac{\partial L(t)}{\partial t}.$$

By a formal integration of the previous expression with $L(0)=0$ we have

$$\frac{t}{L(t)-1} = \ln(1-L(t))$$

which implies the final result. □

This equation for the generating function $L(t)$ allows to express it in terms of the generating function $T(t)$ of unordered trees, which is known to verify the following equation (see [6] or [3]).

$$T(t) = t\ e^{T(t)} \tag{7}$$

Corollary 3.1. *The generating functions $T(t)$ and $L(t)$ verify the relation*

$$L(t) - 1 = -\frac{1}{e^{T(t)}}.$$

Proof. From Lemma 3.1, we have:

$$-\frac{t}{L(t)-1} = t\ e^{-t/(L(t)-1)}.$$

And then, from equation (7), we can say that:

$$-\frac{t}{L(t)-1} = T(t)$$

which, combined with (7), implies the result. □

In order to perform the last step of the proof of Theorem 1.3, we need the following version of the Lagrange Inversion Formula (see [1, page 65]).

Theorem 3.1. *Let $f(z) = \sum_{i\geq 0} f_i z^i$ be a formal power series with $f_0 \neq 0$, and let $Y(t)$ be the unique formal power series solution of the equation $Y = tf(Y)$. Then the coefficients of $g(Y)$ (for an arbitrary series g) are given by*

$$[t^n]\ g(Y(t)) = \frac{1}{n}[z^{n-1}]\ (f(z))^n\ \frac{\partial g(z)}{\partial z}.$$

We can now end the proof of Theorem 1.3. It suffices to apply the previous result to the equation given in Corollary 3.1, with the correspondences $f(t) = e^t$ and $g(t) = -e^{-t}$. Then, we have

$$\begin{aligned} [t^n]\ (L(t)-1) = &\ \frac{1}{n}[z^{n-1}]\ e^{(n-1)z} \\ = &\ [z^{n-1}]\sum_{k\geq 0}\frac{(n-1)^k z^k}{n\ k!} \\ = &\ \frac{(n-1)^{n-1}}{n!}, \end{aligned}$$

which ends the proof of Theorem 1.3. The proof of Corollary 1.1 is a direct consequence of the relation (4) between G_n and H_n.

Acknowledgment We are indebted to Mireille Bousquet-Mélou for helpful discussions simplifying the second part of this work.

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