A bijective proof for the arborescent form of the multivariable Lagrange inversion formula

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A BIJECTIVE PROOF FOR THE ARBORESCENT FORM OF THE MULTIVARIABLE LAGRANGE INVERSION FORMULA

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ABSTRACT. In [12], Goulden and Kulkarni propose a bijective proof of a form of the Lagrange-Good's multivariable inversion formula [9]. In this paper, we propose a new bijective proof of this formula, simpler than Goulden and Kulkarni's proof and we illustrate the interest of this proof in the enumeration of multisort rooted trees such that the edges partition is given.

1. INTRODUCTION

Lagrange's inversion formula for formal power series is a classical tool in enumerative combinatorics (see for example [11, 2]). In 1960, Good [9] proposed an extension that could handle the case of multivariable formal power series, called the Lagrange-Good's formula, and is well adapted for the enumeration of multisort (or multicolored) structures. It has been extensively used by Goulden and Jackson [13, 10, 11]. Let $d \geq 1$ be a fixed integer, $\mathbf{x} = (x_1, \ldots, x_d)$ be a vector of formal variables, $\mathbf{n} = (n_1, \ldots, n_d)$ a vector of integers and let $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$. For a given multivariable formal power series $h(\mathbf{x})$, the coefficient of $\mathbf{x}^{\mathbf{n}}$ in $h(\mathbf{x})$ is denoted by $[\mathbf{x}^{\mathbf{n}}]\mathbf{h}(\mathbf{x})$. There are several forms of Lagrange-Good's formula, and we give the most classical in the following theorem.

Theorem 1. *Let $F(\mathbf{x})$, $R_1(\mathbf{x}), \ldots, R_d(\mathbf{x})$ be $d+1$ multivariable formal power series such that $R_i(\mathbf{0}) \neq 0$ for $i \in [d]$, $\mathbf{0} = (0, \ldots, 0)$ and $A_i(\mathbf{t}) = t_i R_i(\mathbf{A}(\mathbf{t}))$ for $i \in [d]$. Then we have the implicit form of Lagrange-Good's formula*

$$[\mathbf{t}^{\mathbf{n}}] \frac{F(\mathbf{A}(\mathbf{t}))}{\det\left(\delta_{i,j} - t_i \dfrac{\partial R_i(\mathbf{t})}{\partial t_j}\right)_{d\times d}} = [\mathbf{x}^{\mathbf{n}}] F(\mathbf{x}) \mathbf{R}^{\mathbf{n}}(\mathbf{x}), \tag{1}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise, and the explicit form

$$[\mathbf{t}^{\mathbf{n}}] F(\mathbf{A}(\mathbf{t})) = [\mathbf{x}^{\mathbf{n}}] F(\mathbf{x}) \mathbf{R}^{\mathbf{n}}(\mathbf{x}) \det\left(\delta_{i,j} - \frac{x_i}{R_i(\mathbf{x})} \frac{\partial R_i(\mathbf{x})}{\partial x_j}\right)_{d\times d}. \tag{2}$$

In [8] Gessel gives a bijective proof of formula (1). This proof is also given in the terms of the theory of combinatorial species in [2, Section 3.2], where it appears as a natural generalization of the proof of the Lagrange's inversion formula in one variable due to Labelle [15]. In [7], Ehrenborg and Méndez give a combinatorial proof of (2) in the case of formal power series in an infinity of variables, in which the determinant in the right-hand-side of (2) is computed using an involution. The first direct bijective proof is due to Goulden and Kulkarni [12]. In fact, Goulden and Kulkarni show that formula (2) is equivalent to a form of Lagrange-Good's inversion which can be called *arborescent*, based on the notion of derivative according to a rooted tree (which can also be called *arborescence*).

Definition 1. Let $\mathcal{G}$ be a directed graph having $S = \{1, \ldots, d\}$ as set of vertices and A as set of (directed) edges, $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ two vectors of formal variables and formal power series indexed by S. We define the derivative of $\mathbf{f}(\mathbf{x})$ according to $\mathcal{G}$ by

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathcal{G}} = \prod_{j \in S} \left\{ \left(\prod_{(i,j)\in A} \frac{\partial}{\partial x_i} \right) f_j(\mathbf{x}) \right\},$$

where (i, j) stands for an edge directed from the vertex i to the vertex j.

Theorem 2. *Let $F(\mathbf{x})$, $R_1(\mathbf{x}), \ldots, R_d(\mathbf{x})$ be $d+1$ formal power series such that $R_i(\mathbf{0}) \neq 0$ for $i \in [d]$ and $A_i(\mathbf{t}) = t_i R_i(\mathbf{A}(\mathbf{t}))$ for $i \in [d]$. Then we have*

$$[\mathbf{t}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{t})) = \left(\prod_{i=1}^{d} \frac{1}{n_i}\right) [\mathbf{x}^{\mathbf{n-1}}] \sum_{\mathcal{T}} \frac{\partial(F, R_1^{n_1}, \ldots, R_d^{n_d})}{\partial \mathcal{T}}, \tag{3}$$

where $\mathbf{n-1} = (n_1 - 1, \ldots, n_d - 1)$, the sum being taken over all rooted trees $\mathcal{T}$ having $\{0, 1, \ldots, d\}$ as set of vertices, rooted in 0, and in which all edges are directed towards 0.

Goulden and Kulkarni prove this formula by establishing a bijection between rooted trees on d sorts (or d colors) and some endofunctions on d sorts, which is an extension of the classical bijection between "vertebrates"and endofunctions [14, 15]. This arborescent form of Lagrange-Good's inversion formula has been independently discovered by Bender and Richmond [1], who notice that the determinant appearing in the right-hand-side of (2) can cause problems if one wants to obtain asymptotical informations about the coefficients in the A_i's. On the other hand, the arborescent formulation (3) contains only positive terms and therefore is better suited for an asymptotical analysis.

The interest of bijective proofs for Lagrange-Good's inversion formula is at least twofold. First, they give a better understanding of the combinatorial signification of the enumerative formulas that one obtains when applying them. Second, a bijection usually leads to an algorithm for random or exhaustive generation. This then justifies the research for simple and efficient proofs of Lagrange-Good's formula (at least from an algorithmic point of view).

In this paper, we first propose a new bijective proof of formula (3). The bijection that we describe here is based on the same principle as Goulden and Kulkarni's bijection [12], namely the manipulation of multisort rooted trees and endofunctions, and has similar properties from an algorithmic point of view. However, it has two advantages: it is easier to put into application and the proof of its validity is immediate. In a second step, we show that this bijection, combined with the Matrix-Tree Theorem, leads to a combinatorial interpretation of various enumeration formulas for rooted trees with a given edges partition, and also, a version of Lagrange-Good's formula due to Goulden and Jackson [13] (see also [11, Section1.2.13]).

2. A bijective proof of formula (3)

In order to prove (3), we work in the context of the theory of multisort species on totally ordered sets (see [2, Chapter 3]). In a first step, we slightly modify formula (3) in order to take into account the fact that we are manipulating labelled structures and exponential generating functions. If we denote $\mathbf{n}! = \prod_{i=1}^{d} n_i!$, then we obtain the following formula, which is equivalent to (3),

$$\mathbf{n}![\mathbf{t}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{t})) = (\mathbf{n-1})![\mathbf{x}^{\mathbf{n-1}}] \sum_{\mathcal{T}} \frac{\partial(F, R_1^{n_1}, \ldots, R_d^{n_d})}{\partial \mathcal{T}}. \tag{4}$$

2.1. **Multisort rooted trees and endofunctions.** We now give a combinatorial interpretation of both sides of this identity. First, we introduce for $i \in [d]$, the species X_i of singletons of sort i, a vector $\mathbf{R} = (R_1, \ldots, R_d)$ of species on d sorts having generating exponential functions $(R_1(\mathbf{x}), \ldots, R_d(\mathbf{x}))$, and a species F on d sorts having generating exponential function $F(\mathbf{x})$. Using F and $\mathbf{R}$ we construct two new species $F(\mathbf{A_R})$ and $End^{\mathcal{R}}_{F,\mathbf{R}}$.

Definition 2. We denote by $A_{\mathbf{R},i}$ the species of $\mathbf{R}$-enriched rooted trees (on d sorts): for $j \in [d]$, there is an R_j-structure on the fiber (the set of sons) of a vertex of sort j, and the root is of sort i. These species hence verify the functional equations $A_{\mathbf{R},i} = X_i R_i(\mathbf{A_R})$, which is the translation, in terms of species, of the equations $A_i(\mathbf{t}) = t_i R_i(\mathbf{A}(\mathbf{t}))$ of the Lagrange-Good's formula.

The species $F(\mathbf{A_R})$ of $(F, \mathbf{R})$-enriched rooted trees is the species of F-assemblies of $A_{\mathbf{R},i}$-structures. The set of $F(\mathbf{A_R})$-structures having n_i elements of sort i, for $i \in [d]$, is denoted by $F(\mathbf{A_R})[\mathbf{n}]$.

In the rest of this paper, a $(F, \mathbf{R})$-enriched rooted tree over $[\mathbf{n}]$ will be represented as a rooted tree on $\{0\} \cup [\mathbf{n}]$, rooted at 0, such that there is a F-structure on the fiber of 0. We denote by k_i the k^{th} element of sort i. Figure 1 gives an example of a $(F, \mathbf{R})$-enriched rooted tree over $[\mathbf{n}] = [4, 3, 1]$. By convention, all edges in a rooted tree are directed towards the root and the enrichment on the fiber of an element (which is

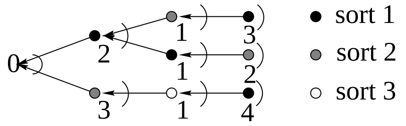

FIGURE 1. A $F(\mathbf{A_R})$-structure.

determinated by its sort) is represented by an arc of a circle. For example the fiber of 2_1 is the set $\{1_1, 1_2\}$ and is embedded into a R_1-structure, while the fiber of 0 is embedded into a F-structure.

By recalling that, for a species G on d sorts, $\mathbf{n}![\mathbf{x^n}]G(\mathbf{x}) = |G[\mathbf{n}]|$, we can interpret the left-hand-side of (4) in the following way.

Lemma 1.

$$\mathbf{n}![\mathbf{t^n}]F(\mathbf{A}(\mathbf{t})) = |F(\mathbf{A_R})[\mathbf{n}]|\,.$$

Definition 3. A $(F, \mathbf{R})$-*enriched partial endofunction* over $[\mathbf{n}]$ is a function f of $[\mathbf{n}]$ into $\{0\} \cup [\mathbf{n}]$ in which each fiber $f^{-1}(u)$ is embedded into a R_i-structure if u is of sort i ($i \in [d]$) and into a F-structure if $u = 0$. We denote by $End^{\mathcal{P}}_{F,\mathbf{R}}[\mathbf{n}]$ the set of $(F, \mathbf{R})$-enriched partial endofunctions over $[\mathbf{n}]$.

Definition 4. Let $f \in End^{\mathcal{P}}_{F,\mathbf{R}}[\mathbf{n}]$. We call *graph of sorts* of f, denoted by $G(f)$, the directed graph having $[d] \cup \{0\}$ as set of vertices and having, for $i \in [d]$, a directed edge from i to $j \neq 0$ (resp. from i to 0), if and only if 1_i is in the fiber of an element of sort j (resp. in the fiber of 0).

One usually represents a partial endofunction f by a directed graph: there is a directed edge from u to v if $f(u) = v$. For example, if $d = 3$ and $[\mathbf{n}] = [4, 4, 3]$, the following figure represents a partial endofunction $f \in End^{\mathcal{P}}_{F,\mathbf{R}}[\mathbf{n}]$ and its related graph of sorts $G(f)$.

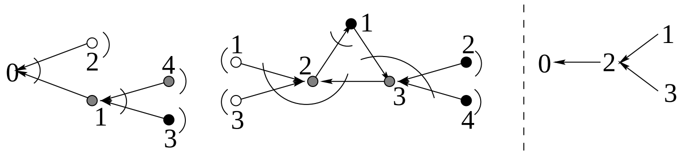

FIGURE 2. A $End^{\mathcal{P}}_{F,\mathbf{R}}$-structure and its graph of sorts.

Definition 5. A partial endofunction $f \in End^{\mathcal{P}}_{F,\mathbf{R}}[\mathbf{n}]$ is said to be *restricted* if $G(f)$ is a rooted tree, rooted at 0, and in which all edges are directed towards the root. We denote by $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$ the set of $(F, \mathbf{R})$-enriched restricted partial endofunctions over $[\mathbf{n}]$.

The partial endofunction given in Figure 2 is therefore a restricted partial endofunction. From now on, unless stated otherwise, we use the terms F-rooted tree for $(F, \mathbf{R})$-enriched rooted tree and endofunction for $(F, \mathbf{R})$-enriched restricted partial endofunction.

The following lemma gives a combinatorial interpretation of the right-hand-side of (4) in terms of endofunctions (see [12, 5] for a proof of this lemma).

Lemma 2.

$$(\mathbf{n}-\mathbf{1})![\mathbf{x^{n-1}}]\sum_{\mathcal{T}} \frac{\partial(F, R_1^{n_1}, \ldots, R_d^{n_d})}{\partial \mathcal{T}} = \left|End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]\right|,$$

the sum being taken over all rooted trees $\mathcal{T}$ having $\{0, 1, \ldots, d\}$ as set of vertices, rooted in 0, and in which all edges are directed towards 0.

We deduce from Lemmas 1 and 2 that in order to prove formula (4) it is sufficient to show that

$$F(\mathbf{A_R})[\mathbf{n}] \simeq End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}],$$

which can be considered as an expression of the arborescent form of Lagrange-Good's inversion formula in terms of species. In the rest of this section, devoted to a bijective proof of this combinatorial identity, which will be illustrated using an example, we call minimal elements of $\{0\} \cup [\mathbf{n}]$ the set $Min([\mathbf{n}]) = \{1_1, 1_2, \ldots, 1_d\}$.

2.2. **From F-rooted trees to endofunctions.** Let $\mathcal{A} \in F(\mathbf{A_R})[\mathbf{n}]$ be a F-rooted tree. We call *skeleton* of $\mathcal{A}$, denoted by $S(\mathcal{A})$, the set of paths starting from an element of $Min([\mathbf{n}])$ and ending at the root 0. $S(\mathcal{A})$ is then a rooted tree having 0 as root and in which all the leaves are in the set $\{1_1, 1_2, \dots, 1_d\}$.

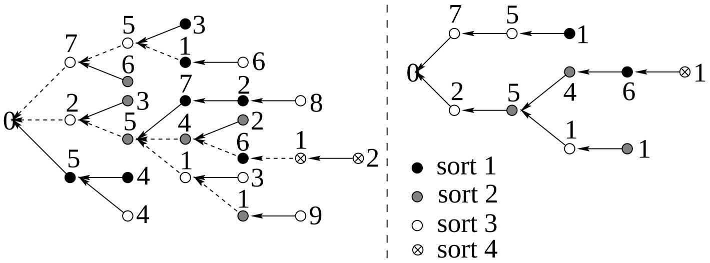

FIGURE 3. A F-rooted tree and its skeleton.

The first step in the transformation of $\mathcal{A}$ into an endofunction $f_\mathcal{A}$ belonging to $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$ consists in isolating $S(\mathcal{A})$. In a second step, for each leaf 1_i in $S(\mathcal{A})$ (covered according to the increasing order of their sort), we denote by C_i the path going from 1_i to the closest ancestor of 1_i belonging to another path C_j with $j < i$ (or to 0 if $i = 1$). In our example,

$$C_1 = 0 \leftarrow 7_3 \leftarrow 5_3 \leftarrow 1_1,$$

$$C_2 = 0 \leftarrow 2_3 \leftarrow 5_2 \leftarrow 1_3 \leftarrow 1_2,$$

$$C_4 = 5_2 \leftarrow 4_2 \leftarrow 6_1 \leftarrow 1_4.$$

In a third step, each path C_i of length l_i is transformed into a biword B_i of length $l_i - 1$ in the following way: for each element u in C_i different from 1_i, if its predecessor (on the path starting from 1_i) is the element k_j, then we perform the transformation

$$u \leftarrow k_j \;\longrightarrow\; \begin{array}{c} *_j \\ \downarrow \\ u \end{array},$$

which indicates that in the fiber of u, k_j will be replaced by an element of sort j, possibly different from k_j. We concatenate the B_i's, which results in a single biword, denoted by $B(\mathcal{A})$:

$$B(\mathcal{A}) = \left(\begin{array}{cccccccccc} *_3 & *_3 & *_1 & *_3 & *_2 & *_3 & *_2 & *_2 & *_1 & *_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 7_3 & 5_3 & 0 & 2_3 & 5_2 & 1_3 & 5_2 & 4_2 & 6_1 \end{array} \right).$$

Finally, for $i \in [d]$, we replace the $*_i$'s in $B(\mathcal{A})$ by the elements of $S(\mathcal{A})$ of sort i taken by increasing order. We then obtain the biword $B'(\mathcal{A})$:

$$B'(\mathcal{A}) = \left(\begin{array}{cccccccccc} 1_3 & 2_3 & 1_1 & 5_3 & 1_2 & 7_3 & 4_2 & 5_2 & 6_1 & 1_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 7_3 & 5_3 & 0 & 2_3 & 5_2 & 1_3 & 5_2 & 4_2 & 6_1 \end{array} \right).$$

From this biword, one can define an endofunction $f_\mathcal{A}$ over $[\mathbf{n}]$ in the following way: for each $u \in [\mathbf{n}]$, if $u \in S(\mathcal{A})$, then $f_\mathcal{A}(u)$ is the image of u in $B'(\mathcal{A})$, otherwise $f_\mathcal{A}(u)$ is the father of u in $\mathcal{A}$. In our example, we obtain the following endofunction $f_\mathcal{A}$.

It remains to show that the partial endofunction $f_\mathcal{A}$ is indeed restricted. We call *biletter* (u, v) an element

$$\begin{array}{c} u \\ \downarrow \\ v \end{array}$$

of $B(\mathcal{A})$ or $B'(\mathcal{A})$ (u can then be of the form $*_j$ or k_j). By construction, we can say that:

- in $B(\mathcal{A})$, at the left of a biletter $(*_i, k_j)$, there always exists a biletter $(*_j, u)$;

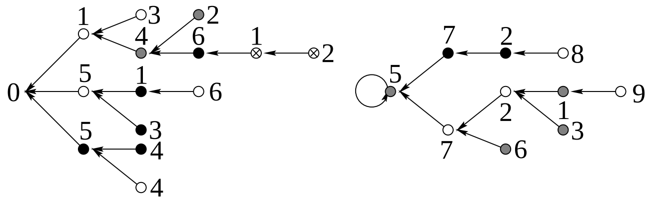

FIGURE 4. The endofunction $f_{\mathcal{A}}$.

- in $B'(\mathcal{A})$, for a biletter $(1_i, u)$, either $u = 0$, or $u = k_j$ and in the latter case, there exists, at the left of $(1_i, u)$, a biletter $(1_j, v)$ (this can be deduced from the previous remark and from the fact that in order to go from $B(\mathcal{A})$ to $B'(\mathcal{A})$, one replaces the $*_i$'s by the elements of $S(\mathcal{A})$ of sort i *taken in increasing order*);
- the first biletter in $B(\mathcal{A})$ is of the form $(1_i, 0)$.

We then deduce that $G(f_{\mathcal{A}})$ is a rooted tree, having root 0 and in which all edges are directed towards 0, which proves that $f_{\mathcal{A}}$ belongs to $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$.

We define the *skeleton* of $f_{\mathcal{A}}$, denoted by $S(f_{\mathcal{A}})$, in the following way: $S(f_{\mathcal{A}})$ includes the elements of all cycles in $f_{\mathcal{A}}$ and all paths starting from elements belonging to $Min([\mathbf{n}])$ and ending either at 0, or in a cycle in $f_{\mathcal{A}}$ ($u \in S(f_{\mathcal{A}})$ if there is $k \geq 1$ such that $f^k_{\mathcal{A}}(u) = u$ or if there are $k \geq 0$ and $i \in [d]$ such that $u = f^k_{\mathcal{A}}(1_i)$).

Property 1. An element $u \notin Min([\mathbf{n}])$ appears k times in $B(\mathcal{A})$ (in the bottom line of $B(\mathcal{A})$) if and only if there exists a maximal subset $\{i_1, \ldots, i_k\}$ of $[d]$ such that for $j = 1, \ldots, k$, $u \in C_{i_j}$.

Property 2. An element u belongs to $S(f_{\mathcal{A}})$ if and only if it belongs to $S(\mathcal{A})$, and its degree $d_{S(f_{\mathcal{A}})}(u)$ in $S(f_{\mathcal{A}})$ is equal to its degree $d_{S(\mathcal{A})}(u)$ in $S(\mathcal{A})$ (in particular, the leaves of $S(f_{\mathcal{A}})$ are the leaves of $S(\mathcal{A})$).

2.3. **From endofunctions to F-rooted trees.** Let $f \in End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$ be an endofunction. The first step of the transformation of f into a F-rooted tree $\mathcal{A}_f$ in $F(\mathbf{A_R})[\mathbf{n}]$ consists in isolating its skeleton $S(f)$. Let f be the endofunction $f_{\mathcal{A}}$ in Figure 4.

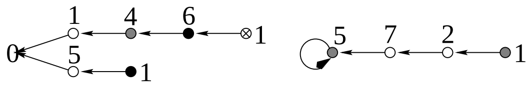

FIGURE 5. The skeleton of f.

We recall that for each element u of $S(f)$, we denote by $d_{S(f)}(u)$ the degree of u in $S(f)$. We then create a set $\mathcal{P}(f)$ and a multiset $\mathcal{M}(f)$:

- $\mathcal{P}(f)$ contains all the elements of $S(f)$ different from 0,
- for each $u \in S(f)$, $\mathcal{M}(f)$ contains k occurrences of u if and only if there are k elements v in $S(f)$ such that $f(v) = u$ (i.e. $d_{S(f)}(u) = k$).

In our example we have $\mathcal{M}(f) = \{0, 0, 6_1, 4_2, 5_2, 5_2, 1_3, 2_3, 5_3, 7_3\}$ and $\mathcal{P}(f) = \{1_1, 6_1, 1_2, 4_2, 5_2, 1_3, 2_3, 5_3, 7_3, 1_4\}$. Now let l be the number of leaves in $S(f)$ and $1_{i_1}, \ldots, 1_{i_l}$ these leaves, with $i_1 < \cdots < i_l$ (here, $l = 3$, $i_1 = 1$, $i_2 = 2$ and $i_3 = 4$). We perform the following transformations on the skeleton of f (the other edges of f are not modified):

- for j from 1 to l, let $u = 1_{i_j}$ and repeat
  (1) let v be the greatest element in $\mathcal{P}(f)$ of the same sort as u,
  (2) suppress in f the edge $f(v) \leftarrow v$ and add the edge $f(v) \leftarrow u$,
  (3) suppress v of $\mathcal{P}(f)$, one occurrence of $f(v)$ in $\mathcal{M}(f)$, and let $u = f(v)$,
  as long as $u \neq 0$ and $u \notin \mathcal{M}(f)$.

We then have to show that the structure $\mathcal{A}_f$ obtained in this way is a F-rooted tree. By definition of the skeleton of an endofunction, and from the fact that the only modifications made on f to obtain $\mathcal{A}_f$ concern its skeleton, it suffices to show that $S(f)$ is transformed into a directed rooted tree with root 0 and in which all edges are directed towards 0. This comes from the fact that for each step $j = 1, \dots, l$ the previous algorithm creates a path starting at 1_{i_j} and ending either at 0 (at least one of the paths created in this way ends at 0), or at an element which also belongs to a path created during a subsequent step (in our example, we create successively the paths $0 \leftarrow 7_3 \leftarrow 5_3 \leftarrow 1_1$, $0 \leftarrow 2_3 \leftarrow 5_2 \leftarrow 1_3 \leftarrow 1_2$ and $5_2 \leftarrow 4_2 \leftarrow 6_1 \leftarrow 1_4$). Finally, it remains to verify that $\mathcal{A}_{f_\mathcal{A}} = \mathcal{A}$ and that $f_{\mathcal{A}_f} = f$. It is a consequence of Properties 1 and 2 and of the following facts:

- during the transformation of $\mathcal{A}$ into $f_\mathcal{A}$, we replace all elements $*_j$ of $B(\mathcal{A})$ by the elements of $S(\mathcal{A})$ of sort j *taken in increasing order*;
- during the transformation of f into $\mathcal{A}_f$, we remove from $\mathcal{P}(f)$ the elements of sort j *in decreasing order*.

By construction, this bijection has the property (also verified in the bijection of Goulden and Kulkarni [12]) that the structure of the fibers of the elements is preserved. We will make use of this property of *fiber structure preservation* in the next section.

Property 3. Let f be an endofunction in $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$ and u any element of $\{0\} \cup [\mathbf{n}]$. For all $i \in [d]$, the restriction of $f^{-1}(u)$ to the elements of sort i and the restriction of the fiber of u in $\mathcal{A}_f$ to the elements of sort i have the same cardinality.

3. Rooted trees with a given edges partition

In this last section, we apply the previous bijection to the enumeration of rooted trees having a given edges partition. The results in this section are already present, implicitly or explicitly, in [13, 10]. However, the simple combinatorial proofs we give here illustrate well the fact that *restricted partial endofunctions are structures that are easier to describe and enumerate than rooted trees*, whence one of the interests of a bijective proof of Lagrange-Good's formula.

Let M be a square matrix of size $(d+1) \times (d+1)$, $M = (m_{i,j})_{0 \le i,j \le d}$, and $\mathcal{A}$ a F-rooted tree over $[\mathbf{n}]$. We say that M is the *edges partition* of $\mathcal{A}$ if there is, in $\mathcal{A}$, $m_{i,j}$ (resp. $m_{i,0}$) edges (recall that all edges are directed towards the root) going from an element of sort i to an element of sort j (resp. to 0). We say that a matrix $M = (m_{i,j})_{0 \le i,j \le d}$ is a *valid edges partition for* $[\mathbf{n}]$ if it verifies that for all $j \in [d]$, $m_{0,j} = 0$, and for all $i \in [d]$, $\sum_{j=0}^{d} m_{i,j} = n_i$. We denote by $\mathbf{m_i}$ the vector corresponding to the i^{th} column $(m_{0,i}, m_{1,i}, \dots, m_{d,i})$ in the matrix M and $\mathbf{M}! = \prod_{i,j} m_{i,j}!$. Moreover, we denote by $F(\mathbf{A_R})[\mathbf{n}]_M$ the set of F-rooted trees over $[\mathbf{n}]$ having M as edges partition. We naturally extend the notion of edges partition to endofunctions and we denote by $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_M$ the set of endofunctions having M as edges partition.

Proposition 1. *Let M be a a valid edges partition for $[\mathbf{n}]$ and $\delta(M)$ the matrix $(\delta_{i,j} n_i - m_{i,j})_{1 \le i,j \le d}$:*

$$|F(\mathbf{A_R})[\mathbf{n}]_M| = \det(\delta(M)) \frac{(\mathbf{n}-\mathbf{1})!}{\mathbf{M}!} \, |F[\mathbf{m_0}]| \prod_{i=1}^{d} |R_i^{n_i}[\mathbf{m_i}]| \, . \tag{5}$$

Proof. By virtue of the bijection between F-rooted trees and endofunctions described in the previous section, and by Property 3 (fiber structure preservation), we have $F(\mathbf{A_R})[\mathbf{n}]_M \simeq End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_M$. In order to give an interpretation to the determinant appearing in the right-hand-side of (5), we use the Matrix-Tree Theorem for directed graphs (see [11, Section 3.3.24] for example):

- let $\lambda = (\lambda_{i,j})_{1 \le i,j \le m}$ the adjacency matrix of a directed labelled graph $\mathcal{G}$ having m vertices: the number of directed spanning trees of $\mathcal{G}$ having root $c \in [m]$ in which all edges are directed towards c is given by
$$\det\left(\left\{\delta_{i,j} \sum_{k=1}^{m} \lambda_{i,k}\right\} - \lambda_{i,j}\right)_{1 \le i,j \le m, i \ne c, j \ne c}.$$

Let M be a valid edges partition for $[\mathbf{n}]$, and $\mathcal{T}$ a rooted tree over $\{0\} \cup [d]$ compatible with M (there cannot be an edge from i to j if $m_{i,j} = 0$). To compute the number of endofunctions f in $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$ having $\mathcal{T}$ as graph of sorts, we note that for $i \in [d]$, if $\mathcal{T}$ has an edge from i to j ($j \in \{0\} \cup [d]$), we have

$$\binom{n_i - 1}{m_{i,0}, \dots, m_{i,j-1}, m_{i,j} - 1, m_{i,j+1}, \dots, m_{i,d}} = \frac{m_{i,j}}{n_i} \binom{n_i}{m_{i,0}, \dots, m_{i,j}, \dots, m_{i,d}}$$

choices for the sort of the images of the elements of sort i different from 1_i, while for 1_i, the sort of its image $f(1_i)$ is fixed by $\mathcal{T}$ (its image is of sort j). We deduce that the number of endofunctions of $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_M$ over $[\mathbf{n}]$ having $\mathcal{T}$ as graph of sorts is

$$\left(\prod_{(i,j)\in\mathcal{T}} m_{i,j}\right)\left(\prod_{i=1}^{d}\frac{1}{n_i}\binom{n_i}{m_{i,0},\ldots,m_{i,d}}\left|R_i^{n_i}[\mathbf{m_i}]\right|\right)\left|F[\mathbf{m_0}]\right|. \tag{6}$$

Now, let T_M be the set of rooted trees over $\{0\}\cup[d]$ rooted at 0 that are compatible with M. By the Matrix-Tree Theorem we can say that $\det(\delta(M)) = \sum_{\mathcal{T}\in T_M}\left(\prod_{(i,j)\in\mathcal{T}} m_{i,j}\right)$, which, combined with (6), let us deduce that

$$\left|End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_M\right| = \det(\delta(M))\left(\prod_{i=1}^{d}\left\{\frac{1}{n_i}\binom{n_i}{m_{i,0},\ldots,m_{i,d}}\left|R_i^{n_i}[\mathbf{m_i}]\right|\right\}\right)\left|F[\mathbf{m_0}]\right|,$$

which implies (5). □

We then immediately deduce a combinatorial proof of a version of Lagrange-Good's formula due to Goulden and Jackson [13] (see also [11, Section1.2.13]) and used, among other, in [13, 10].

Corollary 1. *Let $F(\mathbf{x}), R_1(\mathbf{x}),\ldots,R_d(\mathbf{x})$ be $d+1$ multivariable formal power series such that $R_i(\mathbf{0})\neq 0$ and $A_i(\mathbf{t}) = t_iR_i(\mathbf{A}(\mathbf{t}))$ for $i\in[d]$. Then we have*

$$[\mathbf{t}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{t})) = \left(\prod_{i=1}^{d}\frac{1}{n_i}\right)\sum_M \det(\delta(M))\left(\prod_{i=1}^{d}[\mathbf{x}^{\mathbf{m_i}}]R_i(\mathbf{x})^{n_i}\right)[\mathbf{x}^{\mathbf{m_0}}]F(\mathbf{x}),$$

the sum being taken over all valid edges partitions M for $[\mathbf{n}]$.

A F-rooted tree over $[\mathbf{n}]$ is an ordered (resp. unordered) rooted tree, with root of sort k, if the fiber of 0 only contains one element, of sort k, and if the fiber of every element different from 0 is embedded into a structure of totally ordered set L (resp. unordered set E). A matrix M is a valid edges partition for rooted trees over $[\mathbf{n}]$ having root of sort k if and only if M is a valid edges partition for $[\mathbf{n}]$, $m_{k,0}=1$ and for all $i\in[d]$ different from k, $m_{i,0}=0$. For $i\in[d]$, let us denote $q_j = \sum_{i=1}^{d} m_{i,j}$, and, for a matrix P, $\mathrm{cof}_{k,k}(P)$ the determinant of P where the k^{th} line and the k^{th} column have been deleted.

Corollary 2. *Let M be a valid edges partition for rooted trees over $[\mathbf{n}]$. The number of multisort ordered rooted trees over $[\mathbf{n}]$ having root of sort k and M as edges partition is*

$$\mathrm{cof}_{k,k}(\delta(M))\frac{(\mathbf{n}+\mathbf{q}-\mathbf{1})!}{\mathbf{M}!}. \tag{7}$$

The number of multisort unordered rooted trees over $[\mathbf{n}]$ having root of sort k and M as edges partition is

$$\mathrm{cof}_{k,k}(\delta(M))\frac{\mathbf{n}^{\mathbf{q}}(\mathbf{n}-\mathbf{1})!}{\mathbf{M}!}. \tag{8}$$

Proof. These results are direct consequences of Proposition 1 and of the following remark: by definition, if $\mathcal{A}$ is a rooted tree having root of sort k, in the graph of sorts of $f_{\mathcal{A}}$, the root 0 has exactly one son, and this son is the element 1_k. So by the Matrix-Tree Theorem, we can say that $\det(\delta M) = \mathrm{cof}_{k,k}(\delta(M))$.

Then it suffices to remark that the number of L^{n_i}-structures (we recall that L is the species of totally ordered sets) over $[\mathbf{m_i}]$ is

$$q_i!\binom{n_i+q_i-1}{q_i},$$

to prove (7), and to remark that the number of E^{n_i}-structures (we recall that E is the species of unordered sets) over $[\mathbf{m_i}]$ is $n_i^{q_i}$ to prove (8). □

A matrix $P=(p_{i,j})_{1\le i,j}$ is called the *degrees partition* of a rooted tree $\mathcal{A}$ on $[\mathbf{n}]$ if the number of elements of $\mathcal{A}$ of sort i having degree j (the degree of an element is the number of elements in its fiber) is $p_{i,j}$ (this notion can naturally be extended to the endofunctions). A matrix P is a *valid degrees partitions for rooted trees over* $[\mathbf{n}]$ if it verifies that for $i\in[d]$, $\sum_{j\ge 0} p_{i,j} = n_i$ and $q_i = \sum_{j\ge 0} jp_{i,j}$.

Corollary 3. *Let M be a valid edges partition for rooted trees over $[\mathbf{n}]$ having root of sort k, and P a valid degrees partition for rooted trees over $[\mathbf{n}]$. The number of multisort ordered rooted trees over $[\mathbf{n}]$ having root of sort k, M as edges partition and P as degrees partition is*

$$\mathrm{cof}_{k,k}(\delta(M))\frac{\mathbf{q}!\mathbf{n}!(\mathbf{n}-\mathbf{1})!}{\mathbf{M}!\mathbf{P}!}. \tag{9}$$

The number of multisort unordered rooted trees over $[\mathbf{n}]$ having root of sort k, M as edges partition and P as degrees partition is

$$\mathrm{cof}_{k,k}(\delta(M))\frac{\mathbf{q}!\mathbf{n}!(\mathbf{n}-\mathbf{1})!}{\mathbf{M}!\mathbf{P}!}\prod_{i,j\geq 1}\frac{1}{j!^{p_{i,j}}}. \tag{10}$$

Proof. The proof of these two formulas is similar to the proof of Corollary 2. The only difference is the fact that the number of L^{n_i}-structures on $[\mathbf{m_i}]$ such that $p_{i,j}$ elements have degree j is

$$q_i!\binom{n_i}{p_{i,0},\ p_{i,1},\ p_{i,2},\ldots}.$$

□

By noticing that ordered rooted trees are asymmetric structures, we obtain, from the two previous corollary, the formulas for the corresponding problem for unlabelled ordered rooted trees, by dividing (7) and (9) by $\mathbf{n}!$.

4. Conclusion

In this paper, we presented a simple bijective proof of the arborescent form of the multivariable Lagrange-Good's inversion formula and we illustrated its usefulness in the problem of the enumeration of multisort rooted trees. Using the same kind of reasoning (but with some variations), we are able to give a combinatorial explanation on two formulas enumerating m-ary cacti (a family of planar maps involved in the classification of complex polynomials, see [3] for example) and to deduce from this explanation a uniform random generation algorithm for these structures [4, 5].

However, there is a problem we were not able to solve: design a proof of the arborescent form of the Lagrange-Good's formula involving unlabelled structures and ordinary generating functions. Among all the proofs of the multivariable Lagrange-Good's formula, as far as we know, the only "unlabelled proof" is due to Chottin and is limited to the case of two variables series ($d = 2$) [6]. Such a proof would be of great interest, especially in the design of exhaustive generation algorithms of multisort unlabelled combinatorial structures (like cacti for example). We can think to solve this problem by using the theory of linear species [14]. Indeed, in [14], Joyal gives, in the case of one variable series, an unlabelled analog to the proof of Labelle using linear species. It would be interesting to extend this proof to the multivariable case.

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