Another enumeration of constellations
(Extended abstract)

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ANOTHER ENUMERATION OF CONSTELLATIONS (EXTENDED ABSTRACT)

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ABSTRACT. In this paper we give a formula for the number of m-constellations having n vertices and p faces. We propose a bijective proof of our formula based on two main tools, the matching of Eulerian trees (introduced by Bousquet-Mélou and Schaeffer [3]) and the correspondence between Lagrangian trees and some families of endofunctions [7, 2].

RÉSUMÉ. Dans cet article, nous donnons une formule pour l'énumération des m-constellations ayant n sommets et p faces. La preuve que nous proposons est bijective et repose sur deux ingrédients, la conjugaison d'arborescences Eulériennes [3] et la correspondance entre arborescences Lagrangiennes et certaines familles d'endofonctions [7, 2].

1. Introduction

A *planar map* C is a proper embedding of a connected graph on the sphere such that $v(C) + f(C) - e(C) = 2$, where $v(C)$ (resp. $f(C)$, $e(C)$) is the number of vertices (resp. faces, edges) of C. The degree of a vertex (resp. face) is the number of incidences of edges to this vertex (resp. face). Two maps are isomorphic if there exists an orientation preserving homeomorphism of the sphere that maps vertices (resp. edges, faces) of one of the maps onto vertices (resp. faces, edges) of the other map, and preserves incidences. Here we consider maps up to isomorphism (for more information about combinatorial maps, see for example the work of Cori and Machì [5]).

Definition 1.1. Let $m \geq 2$. An *m-constellation* is a planar map whose faces are colored black and white in such a way that adjacent faces to a given white (resp. black) face are black (resp. white), the degree of any black face is m, and the degree of any white face is a multiple of m. An m-constellation is *rooted* if one of its edges (the root edge) is distinguished.

In the rest of this paper[1], we consider only rooted constellation an we omit the term rooted. We call *polygons* (or *m-gons*) the black faces of an m-constellation. Moreover, we associate to every vertex of an m-constellation a color chosen from a set $\{c_1, \ldots, c_m\}$ of m colors, in such a way that the vertices of any m-gon are colored with $c_1, \ldots, c_m$ in counterclockwise order and the vertices incident to the root edge are colored with c_{m-1} and c_m[2] (see Fig. 1, where the root edge is dashed).

The notion of constellation was introduced and studied from a combinatorial point of view (in relationship with the enumeration of minimal factorizations of permutation) in the thesis of Hanusse [8], and by Bousquet-Mélou and Schaeffer [3]. In this last paper, the authors give bijective proofs for the following results.

Theorem 1.1. [3, Theorem 2.3] *Let $m \geq 2$. The number of m-constellations having d_i white faces of degree mi is*

$$m(m-1)^{f-1} \times \frac{[(m-1)p]!}{[(m-1)p-f-2]!} \times \prod_{i \leq 1} \frac{1}{d_i!} \binom{mi-1}{i-1}^{d_i},$$

where $p = \sum id_i$ and $f = \sum d_i$.

[1] A complete version of this paper is available as a technical report [4]

[2] This convention differs slightly from the paper of Bousquet-Mélou and Schaeffer [3].

Corollary 1.1. [3, Corollary 2.4] *Let $q \geq 1$ and $m \geq 2$. The number of m-constellations formed of q polygons is*
$$\frac{(m+1)m^{q-1}}{[(m-1)q+2][(m-1)n+1]}\binom{mq}{q}.$$

In this paper, we are interested in the enumeration of m-constellations having a given number n of vertices and p of white faces. An m-constellation having only one white face is called an *m-cactus*. Cacti are well known objects [1, 8, 6], and the formula for the number of m-cactus having a given number of vertices is now a classical result (see [1, Theorem 10] for example).

Theorem 1.2. *Let $m \geq 2$ and n such $(m-1)$ divides $(n-1)$ ($(n-1)/(m-1) = q$). The number of m-constellations having n vertices and one white face is*
$$\frac{1}{(m-1)q+1}\binom{mq}{q}.$$

Here, we prove the following result, an extension of Theorem 1.2 to m-constellation having more than one white face.

Theorem 1.3. *Let $m \geq 2$, $n \geq 1$ and $p > 1$ be integers such that $(m-1)$ divides $(n+p-2)$ ($(n+p-2)/(m-1) = q$) and $1 \leq q \leq p$. The number of m-constellations having n vertices and p white faces is*

$$\frac{(m-1)^{p-1}}{n \times p} \times \sum_{k=0}^{q-p} m^{k+1}\binom{q-k-1}{p-1}\binom{q+n-k-2}{n-1}\binom{k+p-2}{p-2}. \tag{1}$$

The principle of our proof is first to reduce the problem to the enumeration of Eulerian trees, by application of the tree matching method introduced in [3] (Section 2), next to reduce this enumeration to the enumeration of some Lagrangian trees, called stellar trees (Section 3), and finally to use a correspondence between trees and endofunctions defined within the context of a bijective proof of the multivariable Lagrange inversion formula [7, 2] to reduce our problem to the enumeration of some enriched endofunctions (Section 4).

2. Constellations and regular Eulerian trees

First, let us define the notion of *regular* constellation: an m-constellation is *regular* if all its white faces have degree m. Regular constellations are related to constellations in the following way (see [3, Proof of Corollary 2.4] and Fig. 1).

Lemma 2.1. *There is a one-to-one correspondence between m-constellations having n_i vertices colored with c_i ($i \in [m]$) and p white faces, and regular $(m+1)$-constellations having n_i vertices colored with c_i ($i \in [m]$) and p vertices colored with c_{m+1}.*

The dual map C^* of a map C describes the incidence relations between the faces of C: the vertices (resp. faces) of C^* are the faces (resp. vertices) of C, and there is an edge of C^* incident to the faces f and f' of C^* if and only if there is an edge incident to the corresponding vertices in C (Fig. 1).

Definition 2.1. A *regular* m-Eulerian map is a *planar bipartite map*[3] such that the degree of any vertex is m.

By duality, there is clearly a one-to-one correspondence between regular m-Eulerian maps and regular m-constellations (by convention, we associate the A-vertices (resp. B-vertices) of an m-Eulerian map to the polygons (resp. white faces) of its dual m-constellation).

[3]A *bipartite map* is an edge-rooted map whose vertices are of two types (A-vertices and B-vertices) such that every edge is incident to an A-vertex and a B-vertex.

Property 2.1. *The coloring of the vertices of a regular m-constellation C induces a coloring of the faces of its dual map E such that:*

1. *the root edge is incident to a face f colored with c_{m-1} and a face f' colored with c_m (following the root edge from its A-vertex to its B-vertex leaves f' on the left and f on the right);*
2. *for every edge e of E, if following e from its A-vertex to its B-vertex leaves on the right (resp. left) a face f (resp. f') colored with c_i (resp. c_j), then $j = (i \mod m) + 1$.*

If we call a coloring of the faces of a regular m-Eulerian map E verifying the previous property a *proper coloring* of E, we can notice that there is exactly one proper coloring of E, determined by its root edge (see Fig. 1, where the A-vertices (resp. B-vertices) are represented by squares (resp. triangles), and the circle associated to a face of the Eulerian map represents its color)

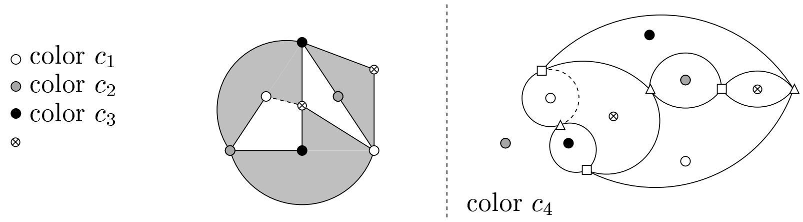


FIGURE 1. A 3-constellation, the corresponding regular 4-constellation and its dual map, a regular 4-Eulerian map

Definition 2.2. A regular m-Eulerian tree is a plane tree such that the vertices are of two types (A-vertices and B-vertices) and every edge is incident to an A-vertex and a B-vertex, the root is an A-leaf (an A-vertex incident to exactly one edge), every internal vertex (a vertex incident to more that one edge) has exactly $(m-1)$ children, and every internal A-vertex has exactly one child which is an internal B-vertex.

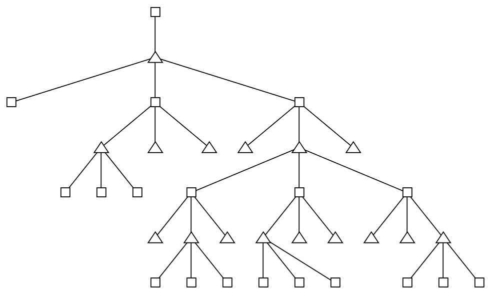


FIGURE 2. A regular 4-Eulerian tree (A-vertices (resp. B-vertices) are represented by squares (resp. triangles))

We recall that a *lr-prefix reading* of a plane tree T reads first the root r of T, then visits recursively the subtrees of T rooted in the children of r, these children being visited from left to right. We call the *matching procedure* of a regular m-Eulerian tree T the following algorithm, which executes two successive lr-prefix readings of T, computing a set of pairs (x, y) of vertices of T such that x is an A-leaf and y a B-leaf (we say that we match x and y, see Fig. 3):

1. let $S = \emptyset$ when the first reading starts;
2. when visiting a B-leaf x for the first time, add x to S;

3. when visiting an unmatched A-leaf y, if $S \neq \emptyset$, then match y with the last visited B-leaf x of S and remove x from S.

We can immediately deduce from the definition of regular m-Eulerian trees that, after this matching, all the B-leaves are matched but there remain m unmatched A-leaves: we call such A-leaves the *single leaves*. A regular m-Eulerian tree T is said to be *balanced* if its root is a single leaf. In the figures that follows, the matched leaves are indicated by a dashed edge and the single leaves are displayed inside a circle.

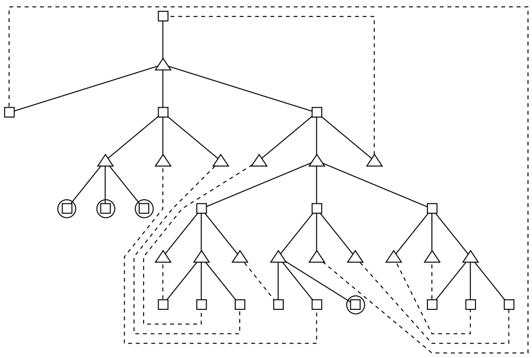


FIGURE 3. Matching of an unbalanced regular 4-Eulerian tree

Property 2.2. [3] *By adding, for every pair (x, y) of matched leaves, an edge incident to x and y, one obtains an unrooted planar bipartite map $M(T)$ such that all the single leaves are incident to the same face of $M(T)$.*

Proposition 2.1. [3] *There is a one-to-one correspondence between balanced regular m-Eulerian trees having p internal A-vertices and q internal B-vertices and regular m-Eulerian maps having $(p+1)$ A-vertices and q B-vertices.*

This result relies on the following construction. Let T be a balanced m-Eulerian tree and $M(T)$ the unrooted planar bipartite map obtained by adding an edge for every pair of matched leaves of T. First, we add, in the face of $M(T)$ containing the single leaves, a plane tree formed of an internal A-vertex and m B-leaves: we call this tree a *star*. Next, for $i \in [m]$, we match the i^{th} B-leaf of the star (visited from right to left) to the i^{th} single leaf of T visited during a lr-prefix reading of T. We denote by $M'(T)$ the unrooted planar bipartite map thus obtained. Finally, for every pair (x, y) of matched leaves, the father of x (resp. y) in T being denoted x' (resp. y'), we remove the vertices x and y (and the edges incident to these vertices) and we add an edge incident to x' and y'. We obtain in this way an unrooted planar bipartite map such that every vertex has degree m. If we root this map at the edge incident to the A-vertex of the star and to the only child of the root of T, we obtain a regular m-Eulerian map (see Fig. 4 for an example and [3] for a description of the inverse construction).

Property 2.3. *Let T be a regular m-Eulerian tree and E the corresponding regular m-Eulerian map. The internal A-vertices (resp. internal B-vertices) of T (including the star) correspond to the A-vertices (resp. B-vertices) of E.*

Property 2.4. *Let T be a regular m-Eulerian tree and E the corresponding regular m-Eulerian map. For every face f of $M'(T)$, there is exactly one edge incident to two matched leaves (an A-leaf x and a B-leaf y) such that following the edge (x, y) from x to y leaves f on the right: we call x the closing A-leaf of f. This closing relation induces a one-to-one correspondence between the A-leaves of T and the faces of E.*

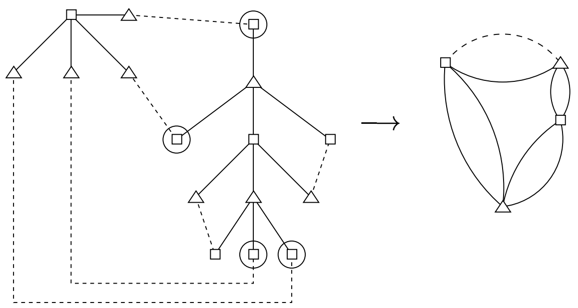


FIGURE 4. A balanced regular 4-Eulerian tree and the related regular 4-Eulerian map

3. Regular Eulerian trees and stellar trees

In this section, we show that the enumeration of m-constellations can be reduced to the enumeration of a family of $(m+1)$-colored trees, called *stellar trees.*

First, we describe an algorithm assigning a color of $\{c_1, \ldots, c_m\}$ to every vertex of a regular m-Eulerian tree T (balanced or unbalanced) in such a way that if T is balanced, the colors of the A-leaves of T are related to the colors of the faces of the corresponding regular m-Eulerian map (Lemma 3.1).

Let T be a regular m-Eulerian tree and $k \in [m]$. The coloring algorithm of T, with c_k for *initial color*, is the following:

1. the root r of T and its child are colored with c_k;
2. during a reading of T, if an A-vertex x other that r (resp. a B-vertex) is colored with c_i, then its $(m-1)$ children are respectively colored, from left to right, with $c_{i+1}, c_{i+2}, \ldots, c_m$, $c_1, c_2, \ldots, c_{i-1}$ (resp. $c_{i-1}, c_{i-2}, \ldots, c_1, c_m, c_{m-1}, \ldots, c_{i+1}$).

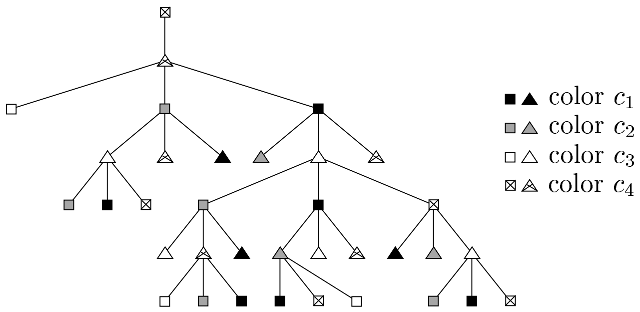


FIGURE 5. A colored regular 4-Eulerian tree, with c_4 for initial color

Lemma 3.1. [4] *Let E be a regular m-Eulerian map and T the corresponding balanced regular m-Eulerian tree, colored with the previous algorithm with c_m for initial color. For every face f of E, the color of f is the same that the color of the A-leaf closing f.*

Lemma 3.2. [4] *Let $\mathbf{n} = (n_1, \ldots, n_m)$ be a vector of positive integers and $a_\mathbf{n}$ (resp. $b_\mathbf{n}$) the number of regular m-Eulerian trees (resp. balanced regular m-Eulerian trees) having n_i A-leaves colored with c_i ($i \in [m]$). Then $a_\mathbf{n} = n_m \times b_\mathbf{n}$.*

We conclude this section by introducing a new family of (Lagrangian) m-colored trees, the m-stellar trees and we show that enumerating regular m-Eulerian trees reduces to enumerating m-stellar trees.

Definition 3.1. Let $m \geq 3$. An m-stellar tree is a plane m-colored tree such that

- the root is colored with c_m,
- the root has $(m-1)$ children, ordered from right to left such that the i^{th} children is colored with c_i;
- for every vertex other that the root, if it is colored with c_i, its children are organized as a totally ordered set of (i,j)-blocks (for an arbitrary $j \in [m]$ such that $i \neq j$), a (i,j)-block being a (unordered) set of $(m-2)$ vertices such that two vertices don't have the same color and no vertex is colored with c_i or c_j.

In the figures that follow, we display blocks as rectangles containing vertices, and we merge the edges linking the vertices of a same block to their father into a single edge.

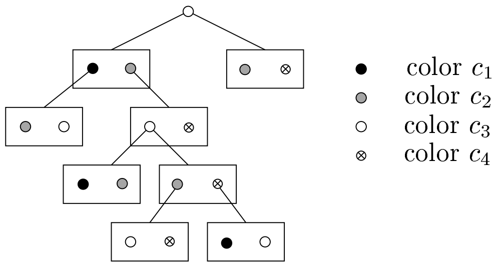


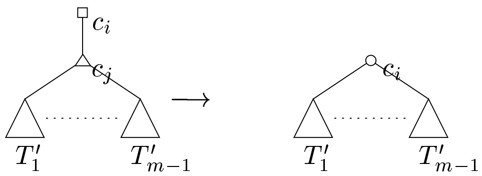
FIGURE 6. A 4-stellar tree

Proposition 3.1. *There is a one-to-one correspondence between regular m-Eulerian trees having n_i A-leaves colored with c_i (for all $i \in [m]$) and m-stellar trees having n_i vertices colored with c_i (for all $i \in [m]$).*

Proof. Let T be a regular m-Eulerian tree having n_i A-leaves colored with c_i ($i \in [m]$). The principle of the transformation of T into an m-stellar tree S is to remove, in three steps, all the internal vertices and the B-leaves of T.

Step 1. During the first step, we remove all the B-leaves of T (we denote by T_1 the tree thus obtained). By definition of a regular m-Eulerian tree, we can say that every internal A-vertex of T_1 has exactly one child, which is an internal B-vertex.

Step 2. During the second step, we remove the B-vertices from T_1 (all of them are internal vertices) by merging every B-vertex x with its father y, the resulting vertex being colored with the color of y. We denote by T_2 the tree thus obtained, which has only vertices of the same type (A-vertices). In the figures that follow, we display the vertices with circles.

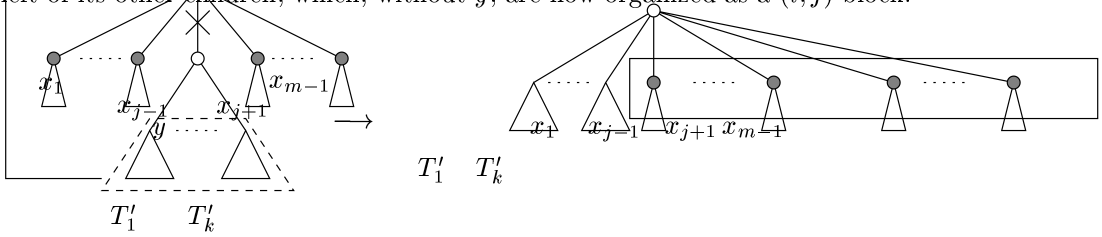


We can deduce from the coloring of T the following properties about T_2.

1. The root r has $(m-1)$ children totally ordered and colored in such a way that the i^{th} rightmost child is colored with c_i.
2. Let $x \neq r$ be a vertex of T_2, colored with c_i (for some $i \in [m]$). There is exactly one $k \in [m]$ ($k \neq i$) such that no child of x is colored with c_k. Moreover, the children of x are totally ordered, from left to right, by cyclically decreasing colors, starting at c_{k-1} $(c_{k-1}, \ldots, c_1, c_m, c_{m-1}, \ldots, c_{k+1})$.

3. The tree T_2 has n_i leaves colored with c_i ($i \in [m-1]$) and $(n_m - 1)$ leaves colored with c_m (due to the fact that the root of T_2 is no more a leaf).

Step 3. Finally, for the last step, we remove the internal vertices from T_2. Let x be a vertex of T_2 other that the root, colored with c_i (for some $i \in [m]$), y the only child of x colored with c_i (see point 2 above) and c_j ($j \neq i$) the color which doesn't appear among the children of x. We remove y from the children of x and we graft its subtrees (the subtrees having for roots the children of y) under x, to the left of its other children, which, without y, are now organized as a (i,j)-block.



Applying recursively this operation to the subtrees of x in the new tree gives an m-stellar tree T' having n_i vertices colored with c_i ($i \in [m-1]$) and n_m vertices colored with c_m.

The reverse construction, proving the one-to-one correspondence follows easily (see [4]). □

4. Skeletons of stellar trees and endofunctions

It follows from the results of the previous sections that enumerating m-constellations having n_i vertices colored with c_i ($i \in [m]$) and p white faces reduces to enumerating $(m+1)$-stellar trees having n_i vertices colored with c_i ($i \in [m]$) and p vertices colored with c_{m+1}. In Theorem 1.3, we consider a less general problem, the enumeration of m-constellation having n vertices and p white faces ($p > 1$), which reduces to the enumeration of $(m+1)$-stellar trees having n vertices colored with a color from $\{c_1, \ldots, c_m\}$ and p vertices colored with c_{m+1}.

In order to enumerate m-stellar trees, we introduce the notion of *skeleton* of an m-stellar tree T (or m-skeleton of T), obtained from T by replacing every vertex colored with c_m by a B-vertex and every other vertex by an A-vertex (Fig. 7).

Definition 4.1. Let $m \geq 3$. An m-skeleton is a plane tree such that

- the vertices are of two types (A-vertices and B-vertices);
- the root is a B-vertex having for children $(m-1)$ totally ordered A-vertices;
- for every A-vertex, its children are organized as a totally ordered set of A-blocks (an A-block being a totally ordered set of $(m-2)$ A-vertices) and B-blocks (a B-block being a totally ordered set of $(m-3)$ A-vertices, followed by one B-vertex);
- for every B-vertex other that the root, its children are organized as a totally ordered set of A-blocks.

As previously, in our figures we represent A-vertices (resp. B-vertices) by squares (resp. triangles) and we merge the edges linking the vertices of a block to their father into a single edge. Moreover, we distinguish four types of edges: an edge incident to the root (resp. to an A-vertex and an A-block; to an A-vertex and a B-block; to a B-vertex and an A-block) is a 1-edge (resp. 2-edge; 3-edge; 4-edge).

Remark 4.1. Let $m \geq 3$, $n \geq 1$ and $p \geq 1$. There exists at least one m-skeleton having n A-vertices and p B-vertices if and only if there exists q such that $q = (n+p-2)/(m-2)$ and $1 \leq q \leq p$. In this case, $(q-1)$ is the number of blocks, $(p-1)$ the number of B-blocks and there are at most $(q-p)$ 4-edges.

Lemma 4.1. [4] *Let $m \geq 3$ and S be an m-skeleton having p B-vertices and k 4-edges. The number of m-stellar trees having S for skeleton is $(m-2)^{p-1}(m-1)^k$.*

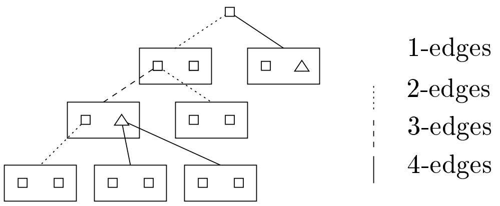


FIGURE 7. A 4-skeleton

Lemma 4.1 implies that in order to prove Theorem 1.3 it is sufficient to enumerate skeletons of Eulerian trees. For this purpose, we introduce a family of endofunctions, called *m-skeleton endofunctions.*

Notation. In Definition 4.2 below, we consider two types of labeled elements called A-elements and B-elements. If the A-elements are labeled on $[n]$ and the B-elements on $[p]$, we denote by $[n,p]$ the union of these two sets of A-elements and B-elements.

Definition 4.2. Let $m \geq 3$. An m-skeleton endofunction over $[n,p]$ is a function g from $[n,p]$ to $\{0\} \cup [n,p]$ such that the preimage $g^{-1}(x)$ of every element of $\{0\} \cup [n,p]$ is organized with the following rules:

1. $g^{-1}(0)$ contains exactly $(m-1)$ A-elements, organized as a totally ordered set, one of them being the A-element labeled 1;
2. for every A-element x, the preimage $g^{-1}(x)$ of x is organized as a totally ordered set of labeled A-blocks (a totally ordered set of $(m-2)$ A-elements) and labeled B-blocks (a totally ordered set of $(m-3)$ A-elements followed by one B-element);
3. for every B-element x, the preimage $g^{-1}(x)$ of x is organized as a totally ordered set of labeled A-blocks.

In the following figure, we represent A-elements (resp. B-elements) by squares (resp. triangles) and an arrow from a block b to an element y means that for every $x \in b$, $g(x) = y$. It follows from this way of representing an endofunction that we can define a notion of edge in an endofunction. Let g be an m-skeleton endofunction. We say that (x,b) is a 1-edge (resp. 2-edge; 3-edge; 4-edge) if $x = 0$ (resp. x is an A-element and b an A-block; x is an A-element and b a B-block; x is a B-element and b an A-block). We use the same convention for drawing edges that in the previous figure.

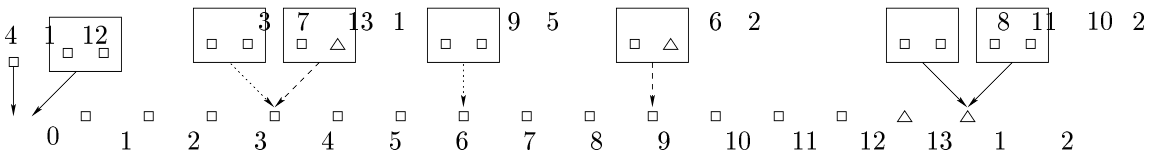


FIGURE 8. A 4-skeleton endofunction over $[13,2]$

Lemma 4.2. *Let $s_{m,n,p,k}$ be the number of m-skeletons having n A-vertices, p B-vertices and k 4-edges ($m \geq 3$, $n \geq 1$, $p \geq 1$, $k \geq 0$), and $e_{m,n,p,k}$ the number of m-skeleton endofunctions over $[n,p]$ having k 4-edges ($m \geq 3$, $n \geq 1$, $p \geq 0$ and $k \geq 0$). Then $e_{m,n,p,k} = n! \times p! \times s_{m,n,p+1,k}$.*

Proof. In the context of describing a bijective proof of a multivariable formal power series inversion formula (called the arborescent form of the multivariable Lagrange formula), there is a one-to-one correspondence between enriched endofunctions and enriched rooted labeled trees (see [7, Theorem

2.1] or [2, Section 2]), which specializes to a one-to-one correspondence between m-skeleton endofunctions over $[n, p]$ and m-skeletons having n A-vertices, $(p+1)$ B-vertices, such that the A-vertices are labeled over $[n]$ and the B-vertices other that the root are labeled over $[p]$, this correspondence conserving the number of 4-edges. The lemma follows from this result and from the fact that an m-skeleton being a plane tree, the number of such labeled m-skeletons is $n! \times p! \times s_{m,n,p+1,k}$. □

Lemma 4.3. *Let $m \geq 3$, $n \geq 1$ and $p > 1$ be integers such that there exists q verifying $q = (n+p-1)/(m-2)$ and $1 \leq q \leq p$. The number of m-skeleton endofunctions having n A-vertices, p B-vertices and k 4-edges is*

$$(m-1) \times (n-1)! \times p! \times \binom{k+p-1}{p-1} \times \binom{q+n-k-2}{n-1} \times \binom{q-1-k}{p}.$$

Proof. First, it follows clearly from the definition of q that there are $(q-1)$ blocks in our endofunctions, p of them being B-blocks (there is exactly one B-element in a B-block and no B-element in the preimage of 0). We deduce from the fact that no B-block is in the preimage of a B-element and from the definition of 4-edges that $0 \leq k \leq (q-1-p)$. Now, we describe the enumeration of these m-skeleton endofunctions in three steps.
Step 1. First we consider the preimage of 0. From the definition of this preimage (point 1 of Definition 4.2), we deduce that 0 can have $(m-1)!\binom{n-1}{m-2}$ different preimages.
Step 2. In a second time, we consider the repartition of the blocks in the preimages of the A-elements and B-elements. We recall that there are $(q-1-p)$ A-blocks and p B-blocks in our endofunctions. Moreover, by fixing the number k of 4-edges, we know that there are exactly k A-blocks in the preimages of the B-elements, which implies that the $(q-1-k)$ other blocks are in the preimages of A-elements. Hence, we have $\binom{k+p-1}{p-1}$ different repartitions of the k blocks in the preimages of the p B-elements, and $\binom{q+n-k-2}{n-1}\binom{q-1-k}{p}$ different repartitions of the $(q-1-k)$ blocks in the preimages of the n A-elements, p of these blocks being B-blocks.
Step 3. During the last step, we assign a label to every element in the blocks. There are $(n-m+1)! \times p!$ different assignments, which finishes the proof. □

Proof of Theorem 1.3. Now we have all the necessary elements allowing to prove our main result. We recall the main steps of our proof.

1. Let $c_{h,i,j}$ be the number of h-constellations having i vertices and j white faces, and $d_{h,i,j}$ be the number of regular h-Eulerian maps having i faces colored with colors from $\{c_1, \dots, c_{h-1}\}$ and j faces colored with c_h. The Lemma 2.1 and the duality relation between constellations and Eulerian maps tells us that $c_{m,n,p} = d_{m+1,n,p}$.
2. Let $r_{h,i,j}$ be the number of regular h-Eulerian trees having i A-leaves colored with colors from $\{c_1, \dots, c_{h-1}\}$ and j A-leaves colored with c_h. From Proposition 2.1, Lemma 3.1, Lemma 3.2 and point 1 above, we have $c_{m,n,p} = r_{m+1,n,p}/p$.
3. Let $t_{h,i,j}$ be the number of h-stellar trees having i A-vertices and j B-vertices. From Proposition 3.1 and point 2 above, we have $c_{m,n,p} = t_{m+1,n,p}/p$.
4. Let $s_{h,i,j,k}$ be the number of h-skeletons having i A-vertices, j B-vertices and k 4-edges. From Remark 4.1, Lemma 4.1, point 3 above and Lemma 4.2, we have

$$c_{m,n,p} = \frac{1}{p} \times \sum_{k=0}^{q-p} \frac{e_{m+1,n,p-1,k}}{n!(p-1)!}(m-1)^{p-1}m^k, \tag{2}$$

   where $q = (n+p-2)/(m-1)$.
5. Finally, (1) follows directly from (2) and Lemma 4.3.

Remark 4.2. For the case $p = 1$ (cacti), the same method, applied with the restriction that the number of 4-edges is always 0, leads to Theorem 1.2.

Bipartite maps. If we replace every A-vertex (resp. B-vertex) of a planar bipartite map by a vertex colored with c_1 (resp. c_2), and every edge by a 2-gon, we obtain a 2-constellation. Hence, plugging $m = 2$ into (1) leads to the following result.

Corollary 4.1. *The number of planar bipartite maps having n vertices and p faces ($p > 1$) is*

$$\frac{2}{n \times p} \times \sum_{k=0}^{n-2} 2^k \binom{n+p-3-k}{p-1} \binom{2n+p-k-4}{n-1} \binom{k+p-2}{p-2}.$$

We can notice that this formula differs from the formula due to Walsh [9]:

$$2 \times \frac{(n+p-3)!}{(p-1)!n!} \times \sum_{k=0}^{n-2} \binom{2n+2p-4}{n-2-k} \binom{p+k}{k}.$$

5. Future work and open questions

The enumeration of constellations is a new topic and it remains many interesting problems to study. Among them, we can think to the enumeration of m-constellations having n_i vertices of color c_i ($i \in [m]$) and p white faces, which is a natural generalization of our main result. In the case of m-cacti, there is a simple formula [6, 1], but we were not able to generalize it to m-constellations, although the Proposition 3.1 (which implies that this problem reduces to the enumeration of stellar trees) seems to be an interesting first step.

As we mentioned in the introduction, the enumeration of constellations is closely related to the enumeration of minimal transitive factorizations of permutations. In this context, the degrees of the vertices and of the white faces is an important parameter. It would be interesting to translate these parameters in terms of stellar trees.

Finally, it is natural to enumerate unrooted constellations. It seems that the method which was applied to the enumeration of unrooted m-cacti [1] and m-constellations [3], can be used with constellations.

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