# Two bijective proofs for the arborescent form of the Good-Lagrange formula and some applications to colored rooted trees and cacti

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# TWO BIJECTIVE PROOFS FOR THE ARBORESCENT FORM OF THE GOOD-LAGRANGE FORMULA AND SOME APPLICATIONS TO COLORED ROOTED TREES AND CACTI

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Abstract. In [19], Goulden and Kulkarni give a bijective proof of an arborescent form of the Good-Lagrange multivariable inversion formula. This formula was first stated explicitly by Bender and Richmond [2] but is implicit in [19]. In this paper, we propose two new simple bijective proofs of this formula and we illustrate the interest of these proofs by applying them to the enumeration and random generation of colored rooted trees and rooted $m$-ary cacti.

## 1. Introduction

The exact enumeration of tree-like structures is a classical area of enumerative combinatorics, beginning at the end of the nineteenth century with the celebrated formula of Cayley for the number of free trees (see for surveys on this topic the books of Moon [26], Goulden and Jackson [17] or Stanley [30]). In this paper we are interested in the enumeration and random generation of colored tree-like structures (trees where vertices are colored). The combinatorial study of these structures (that interested, among others, Tutte [33, 32], Knuth [24], Goulden and Jackson [17] and Bergeron and al. [3]) is strongly related to the Lagrange formula for the inversion of formal power series, and more precisely to the generalizations of this formula to formal power series in several variables. There are different extensions of Lagrange formula in the context of multivariable formal power series (see [14] for a survey on this topic), but one of the most used in enumerative combinatorics was proposed by Good [15] and is known as the Good-Lagrange formula. This formula is particularly well adapted for the enumeration of colored tree-like structures. It has been extensively used by Tutte [33, 32], Goulden and Jackson [22, 16, 17] or Bóna and al. [5] for instance.

Let $m \geq 1$ be a fixed integer, $\mathbf{x} = (x_1, \ldots, x_m)$ be a vector of formal variables, $\mathbf{n} = (n_1, \ldots, n_m)$ a vector of integers and set $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_m^{n_m}$. For a given multivariable formal power series $h(\mathbf{x})$, the coefficient of $\mathbf{x}^{\mathbf{n}}$ in $h(\mathbf{x})$ is denoted by $[\mathbf{x}^{\mathbf{n}}]h(\mathbf{x})$. The set $\{1, 2, \ldots, m\}$ is denoted $[m]$.

**Theorem 1.** (Implicit and explicit forms of Good-Lagrange Formula [15]) *Let $F(\mathbf{x}), R_1(\mathbf{x}), \ldots, R_m(\mathbf{x})$ be given multivariable formal power series and let $A_1(\mathbf{x}), \ldots, A_m(\mathbf{x})$ be formal power series such that $A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x}))$, for $i = 1, \ldots, m$. Then, for any vector $\mathbf{n}$ of non-negative integers we have the implicit form of Good-Lagrange formula*

$$[\mathbf{x}^{\mathbf{n}}]\frac{F(\mathbf{A}(\mathbf{x}))}{\det\left(\delta_{i,j} - x_i \dfrac{\partial R_i}{\partial x_j}(\mathbf{A}(\mathbf{x}))\right)_{m\times m}} = [\mathbf{x}^{\mathbf{n}}]F(\mathbf{x})\mathbf{R}(\mathbf{x})^{\mathbf{n}}, \tag{1}$$

*where $\delta_{i,j} = \chi(i = j)$ and $\mathbf{R}(\mathbf{x})^{\mathbf{n}} = R_1(\mathbf{x})^{n_1} \cdots R_m(\mathbf{x})^{n_m}$, and the explicit form*

$$[\mathbf{x}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{x})) = [\mathbf{x}^{\mathbf{n}}]F(\mathbf{x})\mathbf{R}(\mathbf{x})^{\mathbf{n}} \det\left(\delta_{i,j} - \frac{x_i}{R_i(\mathbf{x})}\frac{\partial R_i(\mathbf{x})}{\partial x_j}\right)_{m\times m}. \tag{2}$$

There have been many combinatorial studies and proofs of these formulas and their variations. We can cite for instance the work related to Rota's theory of polynomial sequences of binomial type (see [21]), the paper of Haiman and Schmitt on incidence algebras [20], and the paper of Gessel [14], where the first bijective proof of the implicit form (1) of Good-Lagrange formula was given. This result was later extended by Zeng [34]. Gessel's proof is a natural generalization of the proof of the Lagrange inversion formula in one variable

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Preliminary versions of some results of this article appeared in [8] and [9]. Most of the work of the second author was done as a Ph.D. student in LaBRI, Université Bordeaux I, under the supervision of Professor Serge Dulucq [10].

due to Labelle [25] and relies on a one-to-one correspondence between some endofunctions and some rooted trees; see also [3, Section 3.2]. Strehl [31] gave also a proof of (1), which, unlike the preceding ones, does not make use of Jacobi's determinantal identity. It is based on the concept of compatible permutations (see also [3, Exercice 3.2.16]). However, most of these works deal with the implicit form of the Good-Lagrange formula, while the most commonly used, at least for the enumeration of tree-like structures, is the explicit form (2). In the case of two variables, a bijective proof was described by Chottin [11] and in the case of an infinite number of variables, Ehrenborg and Méndez [12] gave a combinatorial proof of (2) in which the determinant in the right-hand-side of (2) is computed using an involution. In [19] Goulden and Kulkarni show that formula (2) is equivalent to a form of Good-Lagrange inversion (see (4) below) which can be called *arborescent*, based on the notion of derivative according to a directed graph, and they give a bijective proof of this arborescent form.

**Definition 1.** Let $\mathcal{G}$ be a directed graph having $V = \{0, 1, \ldots, m\}$ as set of vertices and $E \subseteq V \times V$ as set of arcs (= directed edges), with the property that 0 has outdegree[1] $d^+(0)$ equal to 0. Let $\mathbf{f}(\mathbf{x}) = (f_0(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ be a vector of formal power series in $\mathbf{x} = (x_1, \ldots, x_m)$. We define the derivative of $\mathbf{f}(\mathbf{x})$ according to $\mathcal{G}$ by

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathcal{G}} = \prod_{j \in V} \left\{ \left( \prod_{(i,j) \in E} \frac{\partial}{\partial x_i} \right) f_j(\mathbf{x}) \right\}. \tag{3}$$

For example, if $\mathcal{G}$ is the directed graph of Figure 1 then

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathcal{G}} = \left( \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_4} f_0(\mathbf{x}) \right) \left( \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_4} f_1(\mathbf{x}) \right) f_2(\mathbf{x}) \left( \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_4} f_3(\mathbf{x}) \right) f_4(\mathbf{x}).$$

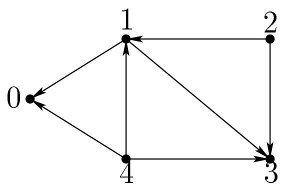


FIGURE 1. A directed graph over the set $\{0, 1, 2, 3, 4\}$ of vertices, with $d^+(0) = 0$.

In the following applications of this concept of derivative with respect to a directed graph, the underlying graph will be a tree $\mathcal{T}$ on the set $V = \{0, 1, \ldots, m\}$, which will be considered as rooted at 0, and where all the edges are directed towards the root. We will denote by $T_m$ the set of these special, implicitly rooted and directed, trees. For $\mathcal{T} \in T_m$, the associated operator $\partial/\partial\mathcal{T}$ on an $m+1$ dimensional vector of series in $\mathbf{x} = (x_1, \ldots, x_m)$ will be called a *tree derivative.*

**Theorem 2.** (Arborescent Good-Lagrange Formula [2], [19].) *Let $F(\mathbf{x}), R_1(\mathbf{x}), \ldots, R_m(\mathbf{x})$ be given multi-variable formal power series and let $A_1(\mathbf{x}), \ldots, A_m(\mathbf{x})$ be formal power series such that $A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x}))$ for $i = 1, \ldots, m$. Then, for any vector $\mathbf{n}$ of non-negative integers we have*

$$[\mathbf{x}^{\mathbf{n}}] F(\mathbf{A}(\mathbf{x})) = \left( \prod_{i=1}^{m} \frac{1}{n_i} \right) [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{\mathcal{T} \in T_m} \frac{\partial (F(\mathbf{x}), R_1(\mathbf{x})^{n_1}, \ldots, R_m(\mathbf{x})^{n_m})}{\partial \mathcal{T}}, \tag{4}$$

*where $\mathbf{n} - \mathbf{1} = (n_1 - 1, \ldots, n_m - 1)$.*

This arborescent form of the Good-Lagrange inversion formula was introduced by Bender and Richmond [2], who noticed that the determinant appearing in the right-hand-side of (2) can cause problems if one wants to obtain asymptotic information about the coefficients in the series $F(\mathbf{A}(\mathbf{x}))$. On the other hand, the arborescent formulation (4) contains only non-negative terms and therefore is better suited to an asymptotical analysis. As we mentioned earlier, it can be retraced to Goulden and Kulkarni, who gave an equivalent form,

[1]The outdegree (resp. indegree) of a vertex in a directed graph is the number of arcs incident to this vertex and directed from (resp. to) this vertex.

based on a summation over rooted trees [19, Theorem 3.1]. Their proof is purely bijective and relies on a correspondence between rooted trees on $m$-colored sets and some endofunctions on these $m$-colored sets, which is an extension of the classical bijection between "vertebrates"and endofunctions [23, 25].

The interest of bijective proofs for the Good-Lagrange inversion formula is at least twofold. First, they give a better understanding of the combinatorial significance of the enumerative formulas that one obtains when applying them. Second, a bijective proof of an enumerative result can very often lead to an algorithm for random or exhaustive generation for the considered objects. This then justifies the search for simple and efficient proofs of Good-Lagrange formula (at least from an algorithmic point of view).

In this paper, we first propose two new bijective proofs of formula (4), both based on a correspondence between colored rooted trees and colored endofunctions. The first one, the simplest, is an extension to the colored case of Prüfer's proof of Cayley's formula enumerating labelled free rooted trees[2]. The second one is based on the same principle as Goulden and Kulkarni's bijection [19], but is somewhat simpler. In the third section, we show that our bijections, combined with the Matrix-Tree Theorem, lead to 1) a combinatorial interpretation of various enumeration formulas for rooted trees with a given arc distribution, 2) a general scheme for random generation of such rooted trees and 3) a combinatorial proof of a version of Good-Lagrange formula due to Goulden and Jackson [22] (see also [17, Section1.2.13]). In the last section we give bijective proofs for two enumerative results about plane cacti, namely the numbers of rooted $m$-ary cacti having a given color distribution and having a given degree distribution, and we deduce from our second proof of formula (4) random generation algorithms for $m$-ary cacti according to these parameters (color distribution and degree distribution).

## 2. Two bijective proofs of Theorem 2

As a first step, we slightly modify formula (4) to work with exponential generating functions and labelled structures. If we denote $\mathbf{n}! = \prod_{i=1}^m n_i!$, then (4) is equivalent to

$$\mathbf{n}![\mathbf{x}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{x})) = (\mathbf{n}-\mathbf{1})![\mathbf{x}^{\mathbf{n}-1}] \sum_{\mathcal{T}\in T_m} \frac{\partial(F(\mathbf{x}), R_1(\mathbf{x})^{n_1}, \ldots, R_m(\mathbf{x})^{n_m})}{\partial\mathcal{T}}. \tag{5}$$

2.1. **Colored linear species.** In order to prove (5), we work in the context of combinatorial species and more precisely of colored (also called multisort) linear species. A linear species is a combinatorial construction on totally ordered sets. We refer the reader to [3, Chap. 5] and [29] for a general treatment of linear species. Let $\ell_1, \ldots, \ell_m$ be a $m$-tuple of disjoint totally ordered sets. We denote by $\ell_1 +_o \cdots +_o \ell_m$ the ordinal sum of these sets, that is the set $\ell_1 \cup \cdots \cup \ell_m$ together with the following total order: $x < y$ if and only if either $x$ and $y$ belong to the same set $\ell_i$ and $x < y$ in $\ell_i$, or $x \in \ell_i$ and $y \in \ell_j$ with $i < j$.

We say that a totally ordered set $\ell$ is $m$-colored if there is a given decomposition $\ell = \ell_1 +_o \cdots +_o \ell_m$. This decomposition is uniquely determined by the multicardinal $\mathbf{n} = (n_1, \ldots, n_m)$ where $n_i = |\ell_i|$, since the $\ell_i$'s form consecutive (possibly empty) segments of $\ell$. We say that $x \in \ell$ has *color* $i$ if $x \in \ell_i$ and the multicardinal $\mathbf{n}$ is called the *color distribution* of $\ell$. There is a canonical $m$-colored set $[\mathbf{n}]$ associated with any vector $\mathbf{n} = (n_1, \ldots, n_m)$ of non-negative integers, namely $[\mathbf{n}] = \ell_1 +_o \cdots +_o \ell_m$, where $\ell_i = [n_i] \times \{i\}$. In other words the elements of the $i^{th}$ component are "colored" by $i$. The pair $(k, i) \in \ell_i$ will sometimes be written more simply $k_i$. Hence, $\ell_i = \emptyset$ if $n_i = 0$ and $\ell_i = \{1_i, \ldots, (n_i)_i\}$ otherwise. We also write $[\mathbf{n}] = [n_1, \ldots, n_m]$ to denote this $m$-colored totally ordered set.

Essentially, an $m$-colored linear species $F$ is a combinatorial rule or construction which assigns, to any $m$-colored totally ordered set $\ell$, a finite set denoted by $F[\ell]$, whose elements are called *$F$-structures on $\ell$*; the only condition imposed on $F$ is, simply stated, that the number $|F[\ell]|$ of $F$-structures on $\ell$ only depends on the multicardinal $(|\ell_1|, \ldots, |\ell_m|)$. In the case of the canonical $m$-colored set $[\mathbf{n}]$ with multicardinality $\mathbf{n} = (n_1, \ldots, n_m)$ described above, we simply write $F[\mathbf{n}]$ instead of $F[[\mathbf{n}]]$.

As usual, we associate with every $m$-colored linear species $F$ the exponential generating function $F(\mathbf{x})$ defined by

$$F(\mathbf{x}) = \sum_{\mathbf{n}} \frac{|F[\mathbf{n}]|}{\mathbf{n}!}\mathbf{x}^{\mathbf{n}}, \tag{6}$$

[2]A rooted tree is called free if there is no structure on the vertex fibers. These trees are sometimes called Cayley trees.

the sum being over all vectors $\mathbf{n}$ of $m$ non-negative integers. We assume that the reader is familiar with the operations on $m$-colored linear species corresponding to the usual operations on formal power series like addition, multiplication or substitution (see [12, 3, 29]). For example, the product $FG$ of two $m$-colored linear species $F$ and $G$ is defined, for any $m$-colored totally ordered set $\ell$, by $(FG)[\ell] = \sum_{\ell'+\ell''=\ell} F[\ell'] \times G[\ell'']$, where the sum is taken over all ordered pairs $(\ell', \ell'')$ of complementary subsets of $\ell$, both of which inherit from $\ell$ a total order and an $m$-coloring. We recall now the definition of partial derivation, which is crucial for the remainder of this paper. Let $X_i$ denote the species of singletons of color $i$, for $i = 1, \ldots, m$. For any $m$-colored species $F$, the partial derivative of $F$ with respect to the color $i$, is a new species, denoted by $\frac{\partial F}{\partial X_i}$, defined, for any $m$-colored totally ordered set $\ell$ by

$$\left(\frac{\partial F}{\partial X_i}\right)[\ell] = F[1 +_i \ell], \tag{7}$$

where $1 +_i \ell$ denotes the totally ordered $m$-colored set obtained from $\ell$ by adding a new minimum element of color $i$. More precisely, if $\ell = \ell_1 +_o \cdots +_o \ell_m$, then $1 +_i \ell = \ell'_1 +_o \cdots +_o \ell'_m$, with $\ell'_i = 1 +_o \ell_i$ and $\ell'_j = \ell_j$ for $j \neq i$; here 1 denotes a one-element set. The exponential generating function of this new species is given by $\frac{\partial F}{\partial X_i}(\mathbf{x}) = \frac{\partial F(\mathbf{x})}{\partial x_i}$.

2.2. **The colored species of rooted trees and endofunctions.** We now give a combinatorial interpretation of both sides of (5). First, we introduce a vector $\mathbf{R} = (R_1, \ldots, R_m)$ of $m$-colored linear species having exponential generating functions $(R_1(\mathbf{x}), \ldots, R_m(\mathbf{x}))$, and an $m$-colored linear species $F$ having exponential generating function $F(\mathbf{x})$. Using $F$ and $\mathbf{R}$ we construct two new $m$-colored linear species $F(A_{\mathbf{R}})$ and $End^{\mathcal{R}}_{F,\mathbf{R}}$, respectively related to the left-hand side and to the right-hand side of (5).

**Terminology.** The set of children of a vertex in a rooted tree is called its *fiber*. This terminology is coherent with the convention, used here, that all rooted trees have their edges implicitly oriented towards the root.

**Definition 2.** Let $\ell$ be an $m$-colored totally ordered set. An $\mathbf{R}$-enriched rooted tree on $\ell$ is a rooted tree on the set $\ell$ of vertices such that for every color $i \in [m]$ and for every vertex of color $i$, there is an $R_i$-structure on its fiber. We denote by $A_{\mathbf{R},i}$ the $m$-colored linear species of $\mathbf{R}$-enriched rooted trees whose root has color $i$.

It follows from this definition that the vector of species $A_{\mathbf{R}} = (A_{\mathbf{R},i}, \ldots, A_{\mathbf{R},m})$ of $\mathbf{R}$-enriched rooted trees verifies the combinatorial equations $A_{\mathbf{R},i} = X_i R_i(A_{\mathbf{R}})$, (for $i = 1, \ldots, m$), which are the translation, in terms of combinatorial species, of the functional equations $A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x}))$ of the Good-Lagrange formula.

We can now consider the composite species $F(A_{\mathbf{R}})$ whose structures on an $m$-colored totally ordered set $\ell$ are $F$-enriched assemblies of $\mathbf{R}$-enriched rooted trees. Notice that for the purpose of the $F$-enrichment, the rooted trees in the assembly can be identified with their roots, yielding the $m$-colored totally ordered subset of $\ell$ on which the $F$-structure is constructed. It is then natural to attach the roots to a new element 0, thus forming a unified rooted tree with root 0, and to consider that the $F$-structure lives on the fiber of this root. These structures will then be called $(F, \mathbf{R})$*-enriched rooted trees.* An example is given in Figure 2 with $\ell = [3, 5, 3, 2]$, where an arc labelled by a species $G$ represents a particular $G$-structure on the corresponding fiber.

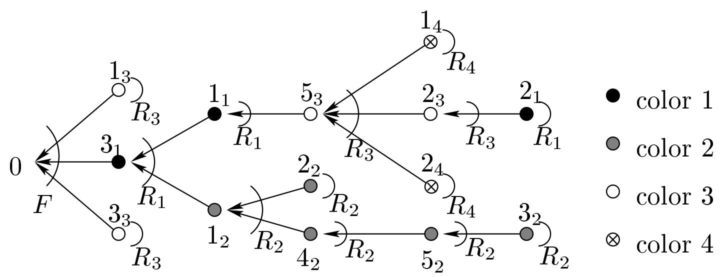


FIGURE 2. An $F(A_{\mathbf{R}})$-structure on $[3, 5, 3, 2]$.

We can then interpret the left-hand-side of (5) as

$$\mathbf{n}![\mathbf{x}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{x})) = |F(A_{\mathbf{R}})[\mathbf{n}]|\,. \tag{8}$$

**Definition 3.** An $(F,\mathbf{R})$-*enriched partial endofunction* on an $m$-colored totally ordered set $\ell$ is a function $f:\ell\to\{0\}\cup\ell$ such that for every color $i\in[m]$ and for every element $u\in\ell_i$, there is an $R_i$-structure on its fiber (also called preimage) $f^{-1}(u)$, and there is a $F$-structure on the fiber of the uncolored element 0. We denote by $End^{\mathcal{P}}_{F,\mathbf{R}}$ the species of $(F,\mathbf{R})$-enriched partial endofunctions.

**Definition 4.** Let $f\in End^{\mathcal{P}}_{F,\mathbf{R}}[\ell]$. We define the *color graph* of $f$, denoted by $G(f)$, as the directed graph having $[m]\cup\{0\}$ as set of vertices and having, for $i=1,\dots,m$, an arc from $i$ to $j\neq 0$ (resp. from $i$ to 0), if and only $f(\min\ell_i)\in\ell_j$ (resp. $f(\min\ell_i)=0$), where $\min\ell_i$ denotes the minimum element of color $i$.

One usually represents a partial endofunction $f$ by its *sagittal graph* (also known as functional digraph), that is a directed graph having $\ell\cup\{0\}$ for vertices and where there is an arc from $u$ to $v$ if $f(u)=v$. For example, with $m=3$ and $\ell=[9,8,6]$, Figure 3 represents a partial endofunction $f$ on $\ell$ and its related color graph $G(f)$.

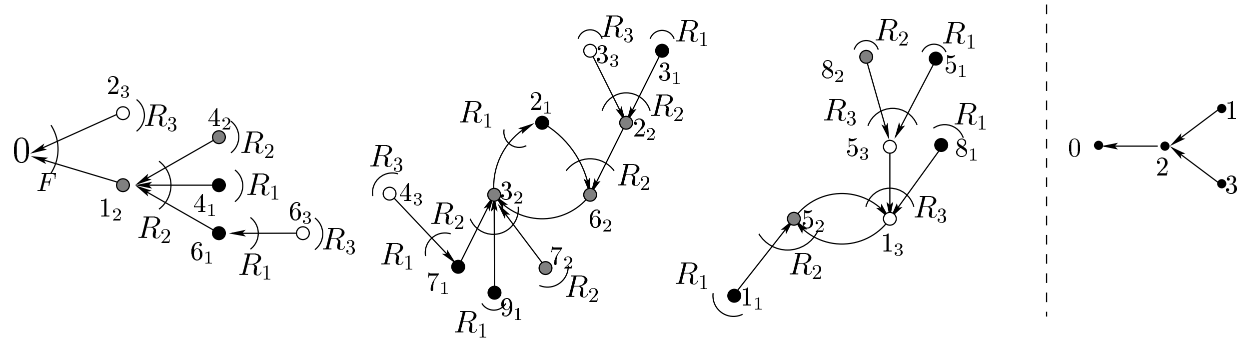


FIGURE 3. An $End^{\mathcal{P}}_{F,\mathbf{R}}$-structure on $[9,8,6]$ and its color graph.

Observe that the sagittal graph of an acyclic $(F,\mathbf{R})$-enriched partial endofunction is nothing but an $(F,\mathbf{R})$-enriched rooted tree. Observe also that the color graph $G(f)$ of a partial endofunction $f$ on $\ell$ is itself the sagittal graph of a partial endofunction $[m]\to\{0\}\cup[m]$. Hence $G(f)$ is acyclic if and only if it is a tree on $V=\{0\}\cup[m]$ with all its edges directed towards 0.

**Definition 5.** A partial endofunction $f\in End^{\mathcal{P}}_{F,\mathbf{R}}[\ell]$ is said to be *restricted* if its color graph $G(f)$ is acyclic. We denote by $End^{\mathcal{R}}_{F,\mathbf{R}}$ the species of $(F,\mathbf{R})$-enriched restricted partial endofunctions.

For example, the partial endofunction given in Figure 3 is restricted, while the $(F,\mathbf{R})$-enriched rooted tree of Figure 2, considered as the sagittal graph of an acyclic $(F,\mathbf{R})$-enriched partial endofunction, is not since the color graph contains a loop at 1. The following lemma gives a combinatorial interpretation of the right-hand-side of (5) in terms of restricted $(F,\mathbf{R})$-enriched partial endofunctions.

**Lemma 1.** *Let $\mathbf{n}=(n_1,\dots,n_m)$ be a vector of non-negative integers. Then*

$$(\mathbf{n}-\mathbf{1})![\mathbf{x}^{\mathbf{n}-\mathbf{1}}]\sum_{\mathcal{T}\in T_m}\frac{\partial(F(\mathbf{x}),R_1(\mathbf{x})^{n_1},\dots,R_m(\mathbf{x})^{n_m})}{\partial\mathcal{T}}=\left|End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]\right|. \tag{9}$$

*Proof.* Let $f$ be an $(F,\mathbf{R})$-enriched partial endofunction (not necessarily restricted) on $[\mathbf{n}]=\ell_1+_o\cdots+_o\ell_m$, with $\ell_i=[n_i]\times\{i\}$. By considering the fibers $f^{-1}(u)$ of the elements $u$ of the list $\{0\}+_o[\mathbf{n}]$, one obtains an $(F\cdot R_1^{n_1}\cdot\ \cdots\ \cdot R_m^{n_m})$-structure on $[\mathbf{n}]$ (see [3, Proposition 3.2.10]). This bijective correspondence leads to the equipotence

$$(F.R_1^{n_1}\cdot\ \cdots\ \cdot R_m^{n_m})[\mathbf{n}]\simeq End^{\mathcal{P}}_{F,\mathbf{R}}[\mathbf{n}]. \tag{10}$$

Let $\mathcal{G}$ be a directed graph on $\{0\}\cup[m]$, with $d^+(0)=0$. For $j\in\{0\}\cup[m]$, we denote by $\mathcal{G}^{-1}(j)$ the set of elements $i$ such that there is an arc from $i$ to $j$ in $\mathcal{G}$. For an $m$-colored linear species $H$, and a vertex $j$ of $\mathcal{G}$, we introduce the derivative of $H$ with respect to $\mathcal{G}^{-1}(j)$ by

$$\frac{\partial H}{\partial\mathcal{G}^{-1}(j)}=\left(\prod_{i\in\mathcal{G}^{-1}(j)}\frac{\partial}{\partial X_i}\right)H. \tag{11}$$

Observe that the derivative of the vector of colored linear species $(F, R_1^{n_1}, \ldots, R_m^{n_m})$ with respect to the directed graph $\mathcal{G}$, as defined by (3), can be expressed as

$$\left(\frac{\partial(F, R_1^{n_1}, \ldots, R_m^{n_m})}{\partial \mathcal{G}}\right) = \left(\frac{\partial F}{\partial \mathcal{G}^{-1}(0)} \cdot \frac{\partial R_1^{n_1}}{\partial \mathcal{G}^{-1}(1)} \cdot \cdots \cdot \frac{\partial R_m^{n_m}}{\partial \mathcal{G}^{-1}(m)}\right). \tag{12}$$

We then make the crucial observation that whenever the (possibly empty) sets $\mathcal{G}^{-1}(0), \mathcal{G}^{-1}(1), \ldots, \mathcal{G}^{-1}(m)$ partition $[m]$, which is equivalent to saying that $\mathcal{G}$ is the sagittal graph of a function $[m] \to \{0\} \cup [m]$, then the set $\left(\frac{\partial(F,R_1^{n_1},\ldots,R_m^{n_m})}{\partial \mathcal{G}}\right)[\mathbf{n-1}]$ can be identified with the set of $(F, \mathbf{R})$-enriched partial endofunctions on $[\mathbf{n}]$ having $\mathcal{G}$ as color graph. By restricting to directed graphs that arise from trees $\mathcal{T} \in T_m$, we obtain that

$$\left(\sum_{\mathcal{T} \in T_m} \frac{\partial(F, R_1^{n_1}, \ldots, R_m^{n_m})}{\partial \mathcal{T}}\right)[\mathbf{n-1}] \simeq End_{F,\mathbf{R}}^{\mathcal{R}}[\mathbf{n}]. \tag{13}$$

This concludes the proof. □

In order to prove formula (5) it is then sufficient to show that for any $m$-colored totally ordered set $\ell$, there is a bijection

$$\Psi : F(A_{\mathbf{R}})[\ell] \tilde{\rightarrow} End_{F,\mathbf{R}}^{\mathcal{R}}[\ell]. \tag{14}$$

This statement can be considered as the combinatorial expression of the arborescent form of Good-Lagrange inversion formula, in terms of $m$-colored linear species.

2.3. **A Prüfer-like bijection.** In this subsection, we present a first construction of (14) based on the classical Prüfer correspondence between free trees and sequences [28]. We refer the reader to [26, 30] for a detailed exposition on this construction and its consequences.

Let $\ell = \ell_1 +_o \cdots +_o \ell_m$ be an $m$-colored totally ordered set and let $T$ be an $(F, \mathbf{R})$-enriched rooted tree on $\ell$. We associate to $T$ an $(F, \mathbf{R})$-enriched restricted partial endofunction $f_T$ on $\ell$ by the following pruning procedure.

1. First identify $T$ with the acyclic $(F, \mathbf{R})$-enriched partial (not necessarily restricted) endofunction whose sagittal graph is $T$.
2. For $i = 1, \ldots, m$, initialize $\mathrm{ind}_i$ as the maximum element of $\ell_i$ and $\sigma_i$ as an empty set of arcs.
3. While $T$ is not reduced to its root 0, let $u$ be its smallest leaf, set $v := T(u)$, and
   (a) remove from $T$ the vertex $u$ and the arc from $u$ to $v$;
   (b) if $u$ is of color $i$, add to $\sigma_i$ the arc $(\mathrm{ind}_i, u)$;
   (c) unless $\mathrm{ind}_i = \min \ell_i$, decrement $\mathrm{ind}_i$ by 1 in $\ell_i$.
4. For $i = 1, \ldots, m$ and for every arc $(u, v)$ of $\sigma_i$, define $f_T$ by replacing $v$ by $u$ in the enriched preimage of $T(v)$. In other words, set $f_T := T \circ \sigma$ where $\sigma$ is the bijection $\ell \to \ell$ induced by the $\sigma_i$'s.

For example, let $T$ be the $(F, \mathbf{R})$-enriched rooted tree on $\ell = [7, 6, 9, 2]$ illustrated in Figure 4. Its smallest

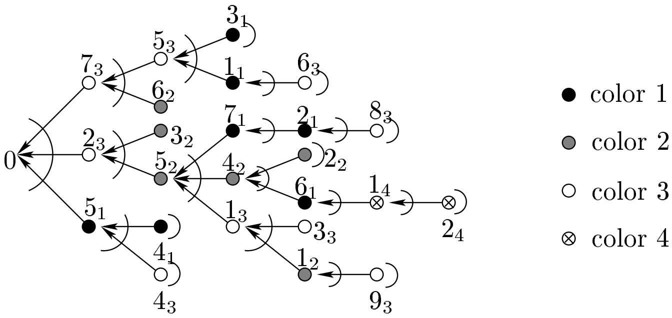


FIGURE 4. An $(F, \mathbf{R})$-enriched rooted tree $T$ on $[7, 6, 9, 2]$.

leaf is the vertex $3_1$, with $T(3_1) = 5_3$. Hence we remove $3_1$ from $T$, and during the phase 3(b) of the algorithm, we add the arc $(7_1, 3_1)$ to $\sigma_1$ (and $3_1$ will be replaced by $7_1$ in $T^{-1}(5_3)$ during step 4). Next, the smallest leaf in the remaining $T$ is the vertex $4_1$, with $T(4_1) = 5_1$. We remove $4_1$ from $T$ and add $(6_1, 4_1)$ to $\sigma_1$. By repeating the same process until $T$ is reduced to its root, we obtain the following lists:

$$\sigma_1 = \{(7_1, 3_1), (6_1, 4_1), (5_1, 5_1), (4_1, 1_1), (3_1, 2_1), (2_1, 7_1), (1_1, 6_1)\},$$
$$\sigma_2 = \{(6_2, 2_2), (5_2, 3_2), (4_2, 6_2), (3_2, 1_2), (2_2, 4_2), (1_2, 5_2)\},$$
$$\sigma_3 = \{(9_3, 3_3), (8_3, 4_3), (7_3, 6_3), (6_3, 5_3), (5_3, 7_3), (4_3, 8_3), (3_3, 9_3), (2_3, 1_3), (1_3, 2_3)\}$$
$$\sigma_4 = \{(1_4, 1_4), (2_4, 2_4)\},$$

and after completing step 4, the $(F, \mathbf{R})$-enriched partial endofunction $f_T$ illustrated in Figure 5.

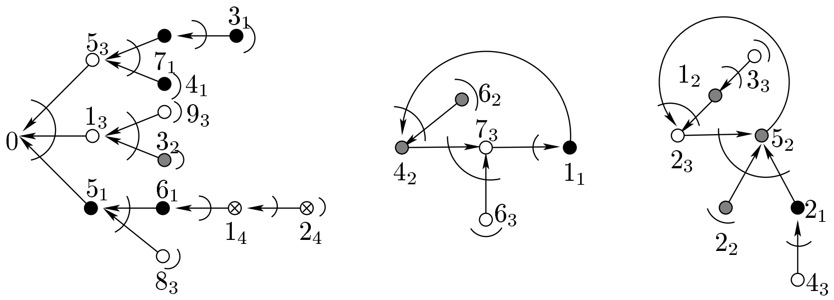


FIGURE 5. The $(F, \mathbf{R})$-enriched restricted partial endofunction $f_T$.

**Claim 1.** The partial endofunction $f_T$ is restricted.

*Proof.* The fact that the color graph $G(f_T)$ is acyclic follows from the following property: for any $i \in [m]$, when the arc $(\min \ell_i,\, u)$ is added to $\sigma_i$, in the step 4(b), $u$ is the last leaf of color $i$ to occur. This implies that no vertex of color $i$ can appear in the image of $f_T$ at this step or later and there can be no cycle in $G(f_T)$. ☐

The reverse map, from an $(F, \mathbf{R})$-enriched restricted partial endofunction $f$, to an $(F, \mathbf{R})$-enriched rooted tree $T_f$, is as follows.

(1) For $i = 1, \ldots, m$, initialize $E_i$ as the list of arcs $(x, y)$ such that $x \in \ell_i$ and $y = f(x)$, ordered by decreasing $x$ ($(x, y) < (x', y')$ if $x > x'$).
(2) Initialize $L$ as the set of leaves of the sagittal graph of $f$ (in other words an element $y$ is a leaf if and only if there are no $i \in [m]$ and $x \in \ell_i$ such that $(x, y) \in E_i$).
(3) For $i = 1, \ldots, m$, initialize $\sigma_i$ as an empty set.
(4) While one of the lists $E_1, \ldots, E_m$ is not empty, let $s_i \in \ell_i$ be the smallest element of $L$ and $(x, y)$ the first element of $E_i$:
   (a) remove $s_i$ from $L$ and $(x, y)$ from $E_i$;
   (b) add the arc $(x, s_i)$ to $\sigma_i$;
   (c) if there remains no arc $(z, y)$, in the $E_j$'s, $j = 1, \ldots, m$, then add $y$ to $L$.
(5) For every arc $(x, s_i)$ of $\sigma_i$ ($i = 1, \ldots, m$), define $T_f$ by replacing $x$ by $s_i$ in the enriched fiber of $f(x)$ (in other words, as a sagittal graph, $T_f$ is $f \circ \sigma^{-1}$).

In the two following claims, we will show that this algorithm is well defined and produces an $(F, \mathbf{R})$-enriched rooted tree.

**Claim 2.** If one of the lists $E_i$ is not empty, then the set $L$ is not empty.

*Proof.* The proof is straightforward by induction on the size of the sets $E_i$. ☐

**Claim 3.** $T_f$ is a $(F, \mathbf{R})$-enriched rooted tree.

*Proof.* It is clear that $T_f$ is the sagittal graph of a partial endofunction and it remains to show that it is acyclic. In order to see this, we observe that during the step 4(b), when we add a pair $(x, s_i)$ $\sigma_i$, we know that $f(x)$ can not have been in $L$ at this time. Thus when we add the arc $(s_i, f(x))$ to $T_f$, $f(x)$ has outdegree 0, which implies that this arc can not create a cycle. ☐

To see that this second algorithm is the converse of the first one it suffices to notice that the bijections $\sigma_i$ constructed in the two algorithms are the same.

2.4. **A Labelle-like bijection.** In this subsection, we denote by $\mathrm{Min}(\ell)$, the set $\{\min \ell_1, \dots, \min \ell_m\}$. We first describe the transformation $\Psi$ from rooted trees to restricted endofunctions.

Let $T \in F(A_{\mathbf{R}})[\ell]$. We call *skeleton* of $T$, denoted by $S(T)$, the set of paths starting from a vertex of $\mathrm{Min}(\ell)$ and ending at the root 0. $S(T)$ is then a sub-rooted tree of $T$ having 0 as root and in which all the leaves are in the set $\mathrm{Min}(\ell)$. For example, the skeleton $S(T)$ of the $(F, \mathbf{R})$-enriched tree of Figure 4 is shown in Figure 6.

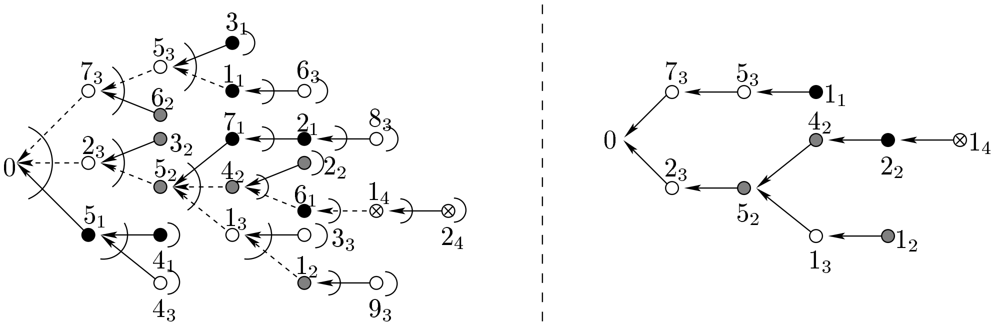


FIGURE 6. An $(F, \mathbf{R})$-enriched rooted tree $T$ and its skeleton $S(T)$.

In a second step, after identifying the skeleton, for each leaf $\min \ell_i$ in $S(T)$ (treated according to the order of their color), we denote by $P_i$ the path $S(T)$ going from $\min \ell_i$ to the closest vertex belonging to another path $P_j$ with $j < i$, or to 0 if such a vertex does not exist. In our example,

$$P_1 = 0 \leftarrow 7_3 \leftarrow 5_3 \leftarrow 1_1,$$

$$P_2 = 0 \leftarrow 2_3 \leftarrow 5_2 \leftarrow 1_3 \leftarrow 1_2,$$

$$P_4 = 5_2 \leftarrow 4_2 \leftarrow 6_1 \leftarrow 1_4.$$

In a third step, each path $P_i$ is transformed into a biword $B_i$ in the following way: for each vertex $x$ in $P_i$ different from $\min \ell_i$, if its predecessor (in $P_i$) is the vertex $k_j \in \ell_j$, then we perform the transformation

$$x \leftarrow k_j \ \mapsto \ \begin{array}{c} *_j \\ \downarrow \\ x \end{array},$$

which indicates that in the fiber of $x$, $k_j$ will be replaced by a vertex of color $j$, possibly different from $k_j$. We concatenate the $B_i$'s, which results in a single biword, denoted by $B(T)$:

$$B(T) = \begin{pmatrix} *_3 & *_3 & *_1 & *_3 & *_2 & *_3 & *_2 & *_2 & *_1 & *_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 7_3 & 5_3 & 0 & 2_3 & 5_2 & 1_3 & 5_2 & 4_2 & 6_1 \end{pmatrix}.$$

Finally, for $i = 1, \dots, m$, we replace the $*_i$'s in $B(T)$ by the vertices of $S(T)$ of color $i$ taken in increasing order. We then obtain the biword $B'(T)$:

$$B'(T) = \begin{pmatrix} 1_3 & 2_3 & 1_1 & 5_3 & 1_2 & 7_3 & 4_2 & 5_2 & 6_1 & 1_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 7_3 & 5_3 & 0 & 2_3 & 5_2 & 1_3 & 5_2 & 4_2 & 6_1 \end{pmatrix}.$$

From this biword, one can define an endofunction $f_T$ on $\ell$ in the following way: for each $x \in \ell$, if $x \in S(T)$, then $f_T(x)$ is the image of $x$ in $B'(T)$, otherwise $f_T(x)$ is the image of $x$ in $T$. In our example, we obtain the endofunction $f_T$ of Figure 7.

**Claim 4.** The color graph of $f_T$ is acyclic.

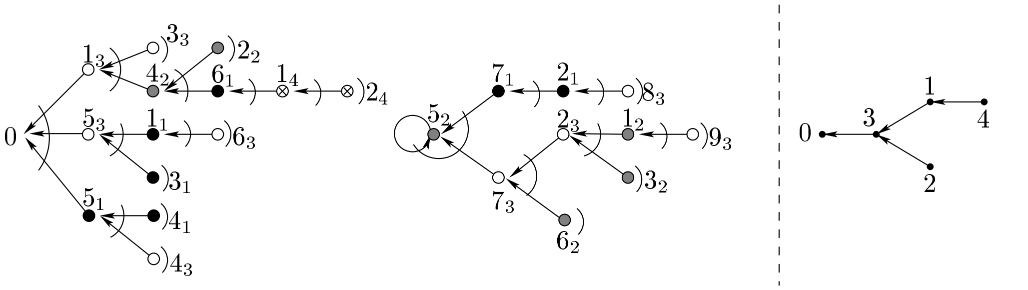


FIGURE 7. The partial endofunction $f_T$ and its color graph.

*Proof.* Observe first that the color graph $G(f_T)$ can be constructed from the biword $B'(T)$. Observe also that by construction, for any $i \in [m]$, the color of $f_T(1_i)$ is either 0 or among colors that have previously occurred, reading $B'(T)$ from left to right. Hence $G(f_T)$ is acyclic $\square$

Recall that the color distribution of a subset $S$ of $\ell$ is the multicardinal $(|S \cap \ell_1|, \ldots, |S \cap \ell_m|)$.

**Property 1.** The transformation $T \to f_T$ preserves the color distribution of the fiber of any element $x \in \{0\} \cup \ell$.

*Proof.* Notice that the transformation $T \to f_T$ only affects the skeleton $S(T)$. Hence it suffices to see that the transformation $S(T) \mapsto B'(T)$ does preserve the color distribution of the fibers, a fact which is obvious, by construction. $\square$

This property then permits us to transport the $(F, \mathbf{R})$-enrichments of the fibers in $T$ into $(F, \mathbf{R})$-enrichments of the corresponding fibers in $f_T$, thus defining the transformation $\Psi : F(A_{\mathbf{R}})[\ell] \to End^{\mathcal{R}}_{F,\mathbf{R}}[\ell]$.

Conversely, given a restricted partial endofunction $f : \ell \to \{0\} \cup \ell$, we define the *skeleton* of $f$, denoted by $S(f)$, in the following way: $S(f)$ includes all the cycles of $f$ and all paths starting from $\mathrm{Min}(\ell)$ and ending either at 0, or on a cycle of $f$. For instance, the skeleton of the endofunction of Figure 7 is shown in Figure 8.

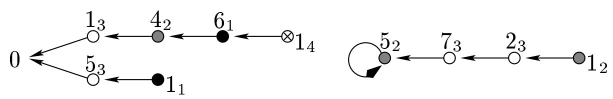


FIGURE 8. The skeleton of $f_T$.

We then create a set $P(f)$ and a multiset $M(f)$ as follows:

- $P(f)$ contains all the vertices of $S(f)$ different from 0,
- for each $x \in S(f)$, $M(f)$ contains $p$ occurrences of $x$ if there are $p$ vertices in the fiber of $x$ in $S(f)$.

In our example we have

$$P(f) = \{1_1, 6_1, 1_2, 4_2, 5_2, 1_3, 2_3, 5_3, 7_3, 1_4\}.$$

and

$$M(f) = \{0, 0, 6_1, 4_2, 5_2, 5_2, 1_3, 2_3, 5_3, 7_3\}$$

Now let $q$ be the number of vertices of $S(f)$ having an empty preimage in $S(f)$ and let $u_1, \ldots, u_q$ be these vertices, in order (here, $u_1 = 1_1$, $u_2 = 1_2$ and $u_3 = 1_4$). We perform the following transformations on the skeleton of $f$ (the other arcs of $f$ are not modified):

(1) for $j$ from $q$ to 1,
  (a) let $x := u_j$;
  (b) while $x \neq 0$ and $x \notin M(f)$
    (i) let $y$ be the greatest vertex in $P(f)$ of the same color as $x$;
    (ii) suppress in $f$ the arc $f(y) \leftarrow y$ and add the arc $f(y) \leftarrow x$;
    (iii) remove $y$ from $P(f)$ and one occurrence of $f(y)$ from $M(f)$;

(iv) let $x := f(y)$,

**Claim 5.** The resulting functional digraph $T_f$ is acyclic.

*Proof.* By definition of the skeleton of an endofunction, and by the fact that the only modifications made on $f$ to obtain $T_f$ concern its skeleton, it suffices to show that $S(f)$ is transformed into a directed rooted tree with root 0 such that all edges are directed towards 0. This comes from the fact that for each step $j = q, \ldots, 1$ the previous algorithm creates a path starting at $u_j$ and ending either at 0 (at least one of the paths created in this way ends at 0), or at a vertex which also belongs to a path created during a subsequent step (in our example, we create successively the paths $5_2 \leftarrow 4_2 \leftarrow 6_1 \leftarrow 1_4$, $0 \leftarrow 2_3 \leftarrow 5_2 \leftarrow 1_3 \leftarrow 1_2$ and $0 \leftarrow 7_3 \leftarrow 5_3 \leftarrow 1_1$). □

**Property 2.** The transformation $f \mapsto T_f$ preserves the color distribution of all fibers of elements $x \in \{0\} \cup \ell$ and thus defines a transformation $End^{\mathcal{R}}_{F,\mathbf{R}}[\ell] \to F(A_{\mathbf{R}})[\ell]$.

Finally, it remains to verify that $T_{f_T} = T$ and that $f_{T_f} = f$. It is a consequence of the fact that during the transformation of $T$ into $f_T$, we replace all vertices $*_j$ of $B(T)$ by the vertices of $S(T)$ of color $j$ taken in increasing order, while during the transformation of $f$ into $T_f$, we remove from $P(f)$ the vertices of color $j$ in decreasing order.

## 3. Enumeration and random generation of colored rooted trees with a given arc distribution

In this section, we apply the previous bijections to the enumeration and random generation of rooted trees having a given arc distribution. Here, the originality resides in the combinatorial nature of the proofs and in the random generation algorithms that these proofs allow. The relative simplicity of these algorithms comes from the facts, expressed in (10) and (13), that partial endofunctions can be seen as lists of fibers (and are then easy to generate) and that in restricted partial endofunction the constraint lies only on the color graph.

Let $P = (p_{i,j})$ be a matrix of size $m \times (m+1)$, with $1 \le i \le m$, $0 \le j \le m$ and let $T$ be an $(F, \mathbf{R})$-enriched rooted tree on $\ell$, viewed as a sagittal graph on $\{0\} \cup \ell$. We consider that the vertex 0 has color 0. We say that $P$ is the *arc distribution* of $T$ if there are, in $T$, $p_{i,j}$ arcs from a vertex of color $i$ to a vertex of color $j$. We say that an $m \times (m+1)$ matrix $P = (p_{i,j})$ is a *valid arc distribution for* $\ell$ if for $i = 1, \ldots, m$, $\sum_{j=0}^{m} p_{i,j} = |\ell_i|$. We denote by $\mathbf{p}_j$ the vector corresponding to the $j^{th}$ column $(p_{1,j}, \ldots, p_{m,j})$ in the matrix $P$. We set $\mathbf{P}! = \prod_{i,j} p_{i,j}!$. Moreover, we denote by $F(A_{\mathbf{R}})[\ell]_P$ the set of $(F, \mathbf{R})$-enriched rooted trees on $\ell$ having $P$ as arc distribution. We naturally extend the notion of arc distribution to endofunctions and we denote by $End^{\mathcal{R}}_{F,\mathbf{R}}[\ell]_P$ the set of restricted partial endofunctions having $P$ as arc distribution.

### 3.1. Enumeration.

**Proposition 1.** *Let $F, R_1, \ldots, R_m$ be given $m$-colored linear species, $\mathbf{n} = (n_1, \ldots, n_m)$ a vector of non-negative integers, and $P$ a valid arc distribution for $\ell = [\mathbf{n}]$. The number of $(F, \mathbf{R})$-enriched rooted trees on $[\mathbf{n}]$ having $P$ as arc distribution is given by*

$$|F(A_{\mathbf{R}})[\mathbf{n}]_P| = \det(\delta(P)) \frac{(\mathbf{n}-\mathbf{1})!}{\mathbf{P}!} |F[\mathbf{p}_0]| \prod_{i=1}^{m} |R_i^{n_i}[\mathbf{p}_i]|, \tag{15}$$

*where $\delta(P)$ is the matrix $(\delta_{i,j} n_i - p_{i,j})_{1 \le i,j \le m}$.*

*Proof.* By virtue of the bijections between rooted trees and partial endofunctions described in the previous section, and by their properties of fiber structure preservation (see Properties 1 and 2), we have $F(A_{\mathbf{R}})[\ell]_P \simeq End^{\mathcal{R}}_{F,\mathbf{R}}[\ell]_P$. We thus proceed to the enumeration of $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_P$.

In order to give an interpretation to the determinant appearing in the right-hand-side of (15), we use the Matrix-Tree Theorem for directed labelled multigraphs (see [4, Theorem 14] for example).

**Matrix-Tree Theorem for directed multigraphs.** Let $\mathcal{G}$ be a directed multigraph on the set $V = \{0\} \cup [m]$ of vertices and let $A = (a_{i,j})_{0 \le i,j \le m}$ be the $(m+1) \times (m+1)$ adjacency matrix of $\mathcal{G}$ ($a_{i,j}$ is the number of arcs from $i$ to $j$). The number of directed spanning trees of $\mathcal{G}$ having root $c \in [m]$ with all arcs directed towards $c$ is equal to

$$\det\left(\left\{\delta_{i,j}\sum_{k=0}^{m} a_{i,k}\right\} - a_{i,j}\right)_{0\le i,j\le m, i\neq c, j\neq c}. \tag{16}$$

Now, let $\mathcal{T}$ be a (free) tree on $\{0\}\cup[m]$ with all arcs directed towards 0. We say that $\mathcal{T}$ is *compatible with* $P$ if $(i,j)\in\mathcal{T}$ implies $p_{i,j}\neq 0$. To compute the number of endofunctions $f$ in $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]$ having $\mathcal{T}$ as color graph, we note that for $i=1,\ldots,m$, if $(i,j)\in\mathcal{T}$, we have

$$\binom{n_i-1}{p_{i,0},\ldots,p_{i,j-1},p_{i,j}-1,p_{i,j+1},\ldots,p_{i,m}} = \frac{p_{i,j}}{n_i}\binom{n_i}{p_{i,0},\ldots,p_{i,j},\ldots,p_{i,m}} \tag{17}$$

choices for the color of the images of the elements of $\ell_i$ other then $1_i := \min(\ell_i)$, while for $1_i$, the color of its image $f(1_i)$ is fixed by $\mathcal{T}$ (its image is of color $j$). It implies that the number of endofunctions in $End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_P$ having $\mathcal{T}$ as color graph is

$$\left(\prod_{(i,j)\in\mathcal{T}} p_{i,j}\right)\left(\prod_{i=1}^{m}\frac{1}{n_i}\binom{n_i}{p_{i,0},\ldots,p_{i,m}}\left|R_i^{n_i}[\mathbf{p}_i]\right|\right)\left|F[\mathbf{p}_0]\right|. \tag{18}$$

Finally, let $T_P$ be the set of trees on $\{0\}\cup[m]$ that are compatible with $P$. By the Matrix-Tree Theorem, with $c=0$, we can say that

$$\sum_{\mathcal{T}\in T_P}\left(\prod_{(i,j)\in\mathcal{T}} p_{i,j}\right) = \det\left(\delta(P)\right), \tag{19}$$

which, combined with (18), yields

$$\left|End^{\mathcal{R}}_{F,\mathbf{R}}[\mathbf{n}]_P\right| = \det\left(\delta(P)\right)\left(\prod_{i=1}^{m}\left\{\frac{1}{n_i}\binom{n_i}{p_{i,0},\ldots,p_{i,m}}\left|R_i^{n_i}[\mathbf{p}_i]\right|\right\}\right)\left|F[\mathbf{p}_0]\right|, \tag{20}$$

which implies (15). □

We then immediately deduce a combinatorial proof of a version of Good-Lagrange formula due to Goulden and Jackson [22] (see also [17, Section1.2.13]).

**Corollary 1.** *Let $F(\mathbf{x}), R_1(\mathbf{x}),\ldots,R_m(\mathbf{x})$ be given multivariable formal power series and let $A_1(\mathbf{x})$, ..., $A_m(\mathbf{x})$ be formal power series such that $A_i(\mathbf{x}) = x_iR_i(\mathbf{A}(\mathbf{x}))$, for $i=1,\ldots,m$. Then for any vector $\mathbf{n}$ of non-negative integers we have*

$$[\mathbf{x}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{x})) = \left(\prod_{i=1}^{m}\frac{1}{n_i}\right)\sum_{P}\det\left(\delta(P)\right)\left(\prod_{i=1}^{m}[\mathbf{x}^{\mathbf{p}_i}]R_i(\mathbf{x})^{n_i}\right)[\mathbf{x}^{\mathbf{p}_0}]F(\mathbf{x}), \tag{21}$$

*the sum being taken over all valid arc distributions $P$ for $[\mathbf{n}]$.*

In the next two results, we apply Proposition 1 to the enumeration of ordered and free $m$-colored rooted trees[3]. This corresponds to taking, for all $i$, $R_i = L$, the $m$-colored species of total orders (lists), and $R_i = E$, of sets, respectively. The corresponding species $A_{\mathbf{R},i}$ of $\mathbf{R}$-enriched rooted trees with root of color $i$ are denoted by $A_{L,i}$ and $A_{E,i}$ (or simply $A_i$), respectively. In order to fit in the form of our previous results, we will augment a rooted tree $T$ on $\ell$ with root of color $i\in[m]$ (called an $i$-rooted tree) to a rooted tree $T_0$ on $\{0\}\cup\ell$ with root 0 by adding one arc from the root of $T$ to 0. In other words, we take $F = X_i$. Note that a *valid arc distribution $P$ for $i$-rooted trees on* $[\mathbf{n}]$ will satisfy $p_{i,0}=1$ and $p_{j,0}=0$ for $j\neq i$. For $j=1,\ldots,m$, let us denote $q_j = \sum_{i=1}^{m} p_{i,j}$, and, for an $m\times m$ matrix $M$, $\mathrm{cof}_{i,i}(M) =$ the determinant of the matrix where the $i^{th}$ line and the $i^{th}$ column have been removed from $M$.

**Proposition 2.** *Let $P$ be a valid arc distribution for $i$-rooted trees on $[\mathbf{n}]$. The number of ordered $m$-colored rooted trees on $[\mathbf{n}]$ having root of color $i\in[m]$ and $P$ as arc distribution is*

$$\mid A_{L,i}[\mathbf{n}]_P \mid = \mathrm{cof}_{i,i}(\delta(P))\frac{(\mathbf{n}+\mathbf{q}-\mathbf{1})!}{\mathbf{P}!}. \tag{22}$$

[3]A ordered rooted tree is a rooted tree such that the fiber of every vertex is enriched with a total order.

*The number of (free) $m$-colored rooted trees on $[\mathbf{n}]$ having root of color $i \in [m]$ and $P$ as arc distribution is*

$$| A_i[\mathbf{n}]_P | = \mathrm{cof}_{i,i}(\delta(P)) \frac{\mathbf{n}^{\mathbf{q}}(\mathbf{n}-\mathbf{1})!}{\mathbf{P}!}. \tag{23}$$

*Proof.* These results are direct consequences of Proposition 1 and of the following remark: if $T_0$ is the augmented tree of an $i$-rooted tree $T$, then in the color graph of the restricted endofunction $f_{T_0}$ obtained by our bijection between enriched rooted trees and enriched partial endofunctions, the root 0 has exactly one son, and this son is the vertex $i$ ($1_i$ is in the fiber of 0). Now, applying the Matrix-Tree Theorem with $V = [m]$ and $c = i$, we can say that $\det(\delta(P)) = \mathrm{cof}_{i,i}(\delta(P))$.

Then it suffices to remark that the number of $L^{n_i}$-structures on $[\mathbf{p}_i]$ is

$$q_i! \binom{n_i + q_i - 1}{q_i},$$

to prove (22), and that the number of $E^{n_i}$-structures on $[\mathbf{p}_i]$ is $n_i^{q_i}$ to prove (23). □

Now, we consider rooted trees having a given arc distribution and a given *degree distribution*. A matrix $D = (d_{i,j})_{1 \le i,j}$ is called the *degree distribution* of a rooted tree $T$ on $\ell$ if the number of vertices of $T$ of color $i$ having indegree $j$ is $d_{i,j}$. This notion can also be applied to endofunctions. Let $P$ be a valid arc distribution for $[\ell]$. A matrix $D$ is a *valid degree distribution for rooted trees on $\ell$ and for $P$* if for $i = 1, \ldots, m$, $\sum_{j \ge 1} d_{i,j} = |\ell_i|$ and $q_i := \sum_{k \ge 1} p_{k_i} = \sum_{j \ge 1} j d_{i,j}$.

**Proposition 3.** *Let $P$ be a valid arc distribution for rooted trees on $[\mathbf{n}]$ having root of color $i \in [m]$, and $D$ a valid degree distribution for rooted trees on $[\mathbf{n}]$ and for $P$. The number of ordered $m$-colored rooted trees on $[\mathbf{n}]$ having root of color $i$, $P$ as arc distribution and $D$ as degree distribution is*

$$| A_{L,i}[\mathbf{n}]_{P,D} | = \mathrm{cof}_{i,i}(\delta(P)) \frac{\mathbf{q}!\mathbf{n}!(\mathbf{n}-\mathbf{1})!}{\mathbf{P}!\mathbf{D}!}. \tag{24}$$

*The number of (free) $m$-colored rooted trees on $[\mathbf{n}]$ having root of color $i$, $P$ as arc distribution and $D$ as degree distribution is*

$$| A_i[\mathbf{n}]_{P,D} | = \mathrm{cof}_{i,i}(\delta(P)) \frac{\mathbf{q}!\mathbf{n}!(\mathbf{n}-\mathbf{1})!}{\mathbf{P}!\mathbf{D}!} \prod_{i,j \ge 1} \frac{1}{j!^{d_{i,j}}}. \tag{25}$$

*Proof.* The proof of these two formulas is similar to the proof of Proposition 2. The only difference is the fact that the number of $L^{n_i}$-structures on $[\mathbf{p}_i]$ such that $d_{i,j}$ vertices have degree $j$ is

$$q_i! \binom{n_i}{d_{i,0},\ d_{i,1},\ d_{i,2}, \ldots}.$$

This gives (24). The proof of (25) then follows immediately. □

**Remark 1.** By noticing that ordered rooted trees are asymmetric structures, we obtain, from the two previous propositions, the formulas for the corresponding problem for unlabelled ordered rooted trees, by dividing (22) and (24) by $\mathbf{n}!$.

3.2. **Random generation.** We can deduce from the proof of Proposition 1 a general scheme for the random generation of $(F, \mathbf{R})$-enriched rooted trees having a given arc distribution $P$, by reducing this problem to the problem of generating at random $F$-structures and products of $R_i$-structures.

**Corollary 2.** *Let $F, R_1, \ldots, R_m$ be given $m$-colored linear species, $\mathbf{n}$ a vector of non-negative integers and $P$ a valid arc distribution for $[\mathbf{n}]$. Given uniform random generation algorithms for $F$-structures and products of $R_i$-structures (for $i = 1, \ldots, m$), it is possible to design a uniform random generation algorithm for $(F, \mathbf{R})$-enriched rooted trees on $[\mathbf{n}]$ having $P$ as arc distribution.*

*Proof.* As in the proof of Proposition 1, let us consider $P$ as the adjacency matrix of a labelled directed multigraph $\mathcal{G}$ on $m+1$ vertices (labelled from 0 to $m$). In order to generate at random a $(F, \mathbf{R})$-enriched rooted tree having $P$ as arc distribution, we can use the following algorithm, which first generates $(F, \mathbf{R})$-enriched partial endofunction having $P$ as arc distribution, and apply the transformation from such endofunctions to rooted trees, which preserves the arc distribution.

(1) First, we generate a random rooted spanning tree $\mathcal{T}_\mathcal{G}$ of $\mathcal{G}$ having root 0 and all the arcs directed towards the root, uniformly among all rooted spanning trees of $\mathcal{G}$ with such characteristics. This can be done in polynomial time and space (see for instance [27], where the algorithms for directed labelled graphs with weighted arcs translate immediately in algorithms for directed labelled multigraphs). This rooted tree will give us the color graph $\mathcal{T}$ of the restricted partial endofunction we will generate.
(2) For $i = 1, \dots, m$, let us denote $j_i = \mathcal{T}(i)$ (the parent of $i$ in $\mathcal{T}$). In a second step, for $i = 1, \dots, m$, we generate a random partition of the $n_i$ elements of color $i$ into $m + 1$ ordered parts (denoted $P_{i,0}, \dots, P_{i,m}$) in such a way that for $k \in \{0\} \cup [m]$, $|P_{i,k}| = p_{i,k}$ and $1_i \in P_{i,j_i}$. This generation problem reduces in fact to the generation of a random permutation of $[n_i] - \{1_i\}$ and can be easily solved. The set $P_{i,k}$, (including $1_i$ if $k = j_i$) will be used as the set of vertices of color $i$ in the fibers of vertices of color $k$ (in the fiber of 0 if $k = 0$).
(3) Now, we generate a random $F$-structure on the set $\cup_{i=0}^m P_{i,0}$, and, for $k = 1, \dots, m$, a random $R_k^{n_k}$-structure on the set $\cup_{i=0}^m P_{i,k}$. By hypothesis, we have algorithms for these random generation problems. The $F$-structure and $R_k^{n_k}$-structures so obtained form a $(F, \mathbf{R})$-enriched partial endofunction having $P$ as arc distribution and $\mathcal{T}$ as color graph.
(4) Finally, we apply one of our transformations from restricted partial endofunctions to rooted trees, which concludes the algorithm.

The fact that this algorithm generates uniformly random $F(A_\mathbf{R})$-enriched rooted trees having $P$ as arc distribution follows directly from the proof of Proposition 1. □

## 4. Enumeration and random generation of rooted $m$-ary cacti

A *cactus* is a connected simple graph in which every 2-connected component is a polygon. An *$m$-cactus* is a cactus in which all polygons have $m$ edges. A *plane $m$-cactus* is an embedding of an $m$-cactus into the plane such that every edge is incident with the unbounded region. An *$m$-ary cactus* is a plane $m$-cactus in which the vertices around each polygon are colored with colors $\{1, 2, \dots, m\}$ in counterclockwise order. Finally, an $m$-ary cactus is called *rooted* if one of its $m$-gon is distinguished; it is equivalent to distinguish one of the edges of the polygon, say from color 1 to color 2.

The $m$-ary cacti form a family of planar maps that appears in different problems, like, for instance, the computation of connection coefficients for the symmetric group [18] or the topological classification of complex polynomials [13]. See [5] for a detailed study of $m$-ary cacti.

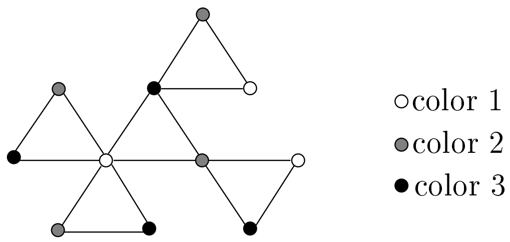


Figure 9. A ternary cacti

The vertex color distribution of an $m$-ary cactus can be described by a vector $\mathbf{n} = (n_1, \dots, n_m)$, where $n_i$ is the number of vertices of color $i$. Figure 9 gives an example of a ternary cactus ($m = 3$) with $\mathbf{n} = (3, 4, 4)$. The *degree* of a vertex is the number of polygons which are incident to it. The vertex degree distribution can be described by an $m \times \infty$ matrix $D = (d_{i,j})_{1 \le i \le m, j \ge 1}$, where $d_{i,j}$ is the number of vertices of color $i$ and of degree $j$. For instance, the vertex degree distribution of the cactus in Figure 9 is given by the following matrix:

$$D = \begin{pmatrix} 2 & 0 & 1 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \cdots \end{pmatrix}.$$

We wish to apply our bijections to the enumeration and the random generation of rooted $m$-ary cacti. Let $\mathcal{K}^\diamond$ denote the $m$-colored species of rooted $m$-ary cacti, where elements of color $i$ are used to label the vertices of color $i$ (such cacti are called labelled cacti), and $\tilde{\mathcal{K}}^\diamond$ the species of rooted (unlabelled) $m$-ary cacti. We first relate $\mathcal{K}^\diamond$ to a special class of $(F, \mathbf{R})$-enriched rooted trees.

For $i = 1, \ldots, m$, we denote by $\widehat{X_i}$ the species $\prod_{j=1, j\neq i}^m X_j$. We call *$i$-block* an $\widehat{X_i}$-structure, and *$m$-cactus-like rooted trees* the $(X_1, L(\widehat{X}_1), \ldots, L(\widehat{X}_m))$-enriched rooted trees, where $L$ denotes the species of total orders. We denote by $A_{\mathcal{K}}$, the species of $m$-cactus-like rooted trees. There is a well known bijection $\Phi : \mathcal{K}^\diamond[\mathbf{n}] \tilde{\rightarrow} A_{\mathcal{K}}[\mathbf{n}]$ (see [5, 6, 7]), as illustrated by Figure 10.

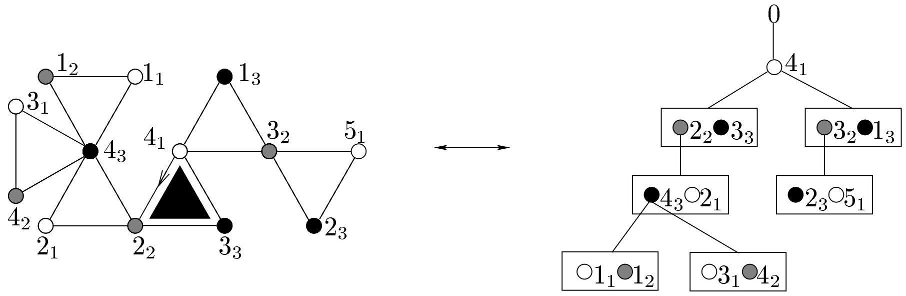

FIGURE 10. A labelled rooted ternary cactus and the corresponding 3-cactus-like tree.

Moreover, this bijection preserves the vertex degree distribution in the following sense.

**Property 3.** With one exception, a vertex $x$ of a rooted $m$-ary cactus $k \in \mathcal{K}^\diamond[\mathbf{n}]$ has degree $d$ if and only if $x$ has exactly $d-1$ blocks in its fiber in the corresponding $m$-cactus-like rooted tree $\Phi(k) \in A_1[\mathbf{n}]$. The exception is the root vertex of color 1, where we have equality.

We can now apply our bijection $\Psi$ with $(X_1, L(\widehat{X}_1), \ldots, L(\widehat{X}_m))$-enriched restricted partial endofunctions. These structures are called *$m$-cactus-like endofunctions*, and the corresponding species is denoted by $m-\mathrm{End}$. Hence, we have

$$\Psi : A_1[\mathbf{n}] \longrightarrow m-\mathrm{End}[\mathbf{n}].$$

Combining with $\Phi$, we see that enumerating $m$-ary cacti can be done by enumerating $m$-cactus-like endofunctions.

Now, we give bijective proofs of two enumerative results about rooted $m$-ary cacti, with respect to the color distribution (Theorem 3) and to the degree distribution (Theorem 4). See [5, Lemma 8 and 9] for necessary and sufficient conditions ensuring that such distributions are valid. We will conclude this section by deducing from these bijections two random generation algorithms for rooted $m$-ary cacti.

### 4.1. Enumeration.

**Theorem 3.** [5, 6] *Let $\mathbf{n}$ be a valid color distribution. The number of rooted $m$-ary cacti having $\mathbf{n}$ as color distribution is given by*

$$\widetilde{\mathcal{K}}^\diamond_{\mathbf{n}} = \frac{1}{b} \prod_{i=1}^m \binom{b}{n_i}, \tag{26}$$

*where $n = \sum_{i=1}^m n_i$, and $b = (n-1)/(m-1)$ is the number of polygons.*

*Proof.* We will enumerate $m$-cactus-like endofunctions having $n_i$ vertices of color $i$. This will give us the number of labelled rooted $m$-ary cacti having $\mathbf{n}$ as color distribution; dividing by $\mathbf{n}!$ will give the result. We enumerate these endofunctions in three steps: 1) counting the number of ways to assign to every vertex the number of blocks in his preimage, 2) counting the number of ways to place the minimum vertex of each color in these blocks in order to have a restricted endofunction, and 3) counting the number of ways to place non minimum vertices in these blocks.

First, in a cactus-like rooted tree $T$, the root 0 has one son of color 1. This implies that the preimage of 0 in the endofunction $\Psi(T) = f_T$ should contain only one vertex, of color 1, namely the element $1_1$. Hence the color tree $G(f)$ of an $m$-cactus-like endofunction $f$ is such that the root 0 has exactly one child, this child being labelled by 1.

From the definition of a valid color distribution $\mathbf{n}$, we can deduce that $b = (n-1)/(m-1)$ is the number of blocks in every endofunction having $\mathbf{n}$ as color distribution, and more precisely that the number of 1-blocks is $b - n_1 + 1$ and the number of $i$-blocks (for $i = 2, \ldots, m$) is $b - n_i$. Let us call a *block partition* of an endofunction the vector of integers $(b_{1,1}, \ldots, b_{1,n_1}, \ldots, b_{m,1}, \ldots, b_{m,n_m})$ where $b_{i,k}$ is the number of $i$-blocks in the preimage of the vertex $k$. If we recall that the number of partitions of a set of $k$ indistinguishable elements in $\ell$ totally ordered parts (that can be empty) is

$$\binom{k+\ell-1}{\ell-1},$$

we can say the the number of possible block partitions for a $m$-cactus-like endofunction having $\mathbf{n}$ as color distribution is

$$\binom{b}{n_1-1}\prod_{i=2}^{m}\binom{b-1}{n_i-1} = \frac{1}{(b-n_1+1)b^{m-1}}\prod_{i=1}^{m} n_i\binom{b}{n_i}. \tag{27}$$

Next, we are interested in the number of ways we can put minimum elements $1_i$, $i = 1, \ldots, m$ in the blocks. We recall that the vertex $1_1$ is not concerned because it is in the preimage of 0. We will show that this number is

$$b^{m-2}(b-n_1+1). \tag{28}$$

To prove this, we proceed as follows: we suppose that the $b$ blocks have been been labelled from 1 to $b$ (in any way) before being assigned to preimages and we use a variant of Prüfer's encoding of the color tree to associate to every different placement of the minimum vertices a word of length $m-1$ on the alphabet $[b]$ such that its last letter is the number (label) of a 1-block.

Indeed, let $\mathcal{T}$ be the color tree related to a placement of minimum vertices of colors other than 1 in blocks (the complete color tree related to the $m$ minimal elements is the tree obtained from $\mathcal{T}$ by adding a parent labelled 0, the new root, to the vertex 1). By definition, this tree has root 1 and by Prüfer's encoding, we can associate to this tree a word $w_{\mathcal{T}}$ of length $m-1$ on the alphabet $[m]$. In order to obtain a word on the alphabet $[b]$, it suffices to modify Prüfer's algorithm in the following way:

- let $W_{\mathcal{T}}$ be the empty word,
- as long as $\mathcal{T}$ is not reduced to its root 1, let $i$ be its lowest leaf and $j$ the parent of $i$: remove the edge $(i, j)$ from $\mathcal{T}$ and add to the end of $w_{\mathcal{T}}$ the label of the $j$-block containing $1_i$.

Hence, we obtain clearly a word of length $m-1$ on $[b]$ such that the last letter is the label of an 1-block. We denote this word by $W_{\mathcal{T}}$. The reverse transformation, relating the placement of minimum vertices in blocks to a word $W$ of length $m-1$ on $[b]$ such that the last letter is the label of an 1-block is the following:

- let $w$ be the word obtained from $W$ by replacing every letter $k$ of $W$ by the letter $j \in [m]$ if and only if in the block partition the block labelled with $k$ is a $j$-block,
- then apply the classical Prüfer's algorithm building from $w$ a free tree $\mathcal{T}_w$ on $[m]$ with root labelled 1, and labelling its edges in increasing order of creation (hence its edges are labelled from 1 to $m-1$),
- for $i = 1, \ldots, m$, $i \neq 1$, if the edge linking $i$ to its parent $j$ in $\mathcal{T}_w$ is labelled with $k$, then put $1_i$ in the $j$-block labelled with $k$.

For instance, let the Figure 11 describe a block partition, with labelled blocks (from 1 to 4), a placement of the minimal elements and the corresponding color tree. The words related to this placement are respectively $W_{\mathcal{T}} = 4.1.2$ and $w_{\mathcal{T}} = 3.1.1$.

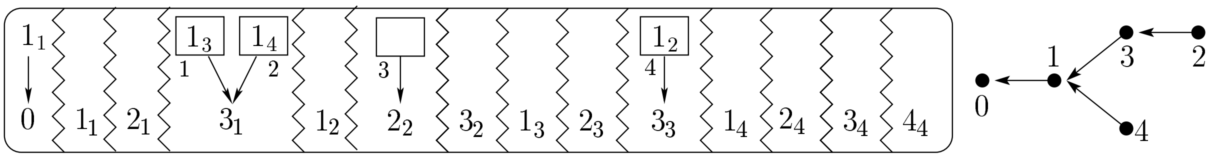


FIGURE 11. A labelled block partition with minimal elements.

Finally, we just have to put non minimum vertices into the blocks, and there are clearly $(\mathbf{n}-\mathbf{1})!$ distinct ways to do that. If we multiply together $(\mathbf{n}-\mathbf{1})!$, (27) and (28), we obtain the number of labelled rooted $m$-ary cacti on $[\mathbf{n}]$, and it suffices to divide by $\mathbf{n}!$ to obtain (26). □

**Theorem 4.** [18, 5] *Let $D=(d_{i,j})_{m\times\infty}$ be a valid degree distribution. The number of rooted $m$-ary cacti having $D$ as degree distribution is given by*

$$\tilde{\mathcal{K}}^{\diamond}_D = b^{m-1}\prod_{i=1}^{m}\frac{1}{n_i}\binom{n_i}{\mathbf{d_i}}, \tag{29}$$

*where $n_i=\sum_{j\geq 1} d_{i,j}$, $n=\sum_{i=1}^m n_i$ and $b=(n-1)/(m-1)$.*

*Proof.* The proof follows the same scheme as in the previous result. If we call *degree* of a vertex of a $m$-cactus-like endofunction the number of blocks in the preimage of this vertex, it follows immediately from Property 3 and the properties of the bijection $\Psi$ that the rooted $m$-ary cacti having $D$ as degree distribution and such that the color 1 vertex of the rooted polygon has degree $l$ is in one-to-one correspondence with the $m$-cactus-like endofunctions having

(a) $d_{i,j}$ vertices of color $i$ and degree $j-1$ if $i>1$ or $i=1$, $j\neq l$ and $j\neq(l+1)$,
(b) $d_{1,l}-1$ vertices of color 1 having degree $l-1$ and
(c) $d_{1,l+1}+1$ vertices of color 1 and degree $l$.

First we are interested in computing the number of block partitions of such endofunctions. For $i=2,\ldots,m$, the number of ways of partitioning $i$-blocks in the fibers of the vertices of color $i$ is

$$\binom{n_i}{d_{i,1},\, d_{i,2},\ldots}.$$

For $i=1$, it follows from (b) and (c) that after the placement of $\sum_{j\geq 1} d_{1,j}$ 1-blocks, there remains one 1-block that has not be placed in the fiber of a color 1 vertex, and the number of different positions where this block can be placed is

$$n_1+\sum_{j\geq 1}(j-1)d_{1,j}-1 = n_1+(b-n_1+1)-1=b.$$

Hence the number of block partitions where a 1-block has been distinguished is

$$b\prod_{i=1}^{m}\binom{n_i}{d_{i,1},\, d_{i,2},\ldots}. \tag{30}$$

Now, we can switch to the number of placements of minimum vertices other than $1_i$, and, in the same way as previously we associate to such a placement a word of length $m-1$ on $[b]$, with the only difference that the last letter of this word is the label of the distinguished 1-block (hence we get rid of the fact that a block was distinguished in the computation of the number of blocks partitions). Hence, we have exactly $b^{m-2}$ such placements of minimum vertices.

Finally, we just have to put non minimum vertices into the blocks, and there are clearly $(\mathbf{n}-\mathbf{1})!$ distinct ways to do that, which leads to our result, after dividing by $\mathbf{n}!$ to forget the vertex labelling. □

**Remark 2.** Recently, Dimitri Zvonkine [35] proposed another combinatorial proof of Theorem 4.

4.2. **Random generation.** The proofs of the previous results are interesting, first because they are the first completely bijective proofs of Theorems 3 and 4, but also because we can immediately deduce from them efficient uniform random generation algorithms for rooted $m$-ary cacti with respect to the color distribution or to the degree distribution. We describe below a random generation algorithm for these two problems.

(1) Input: color (resp. degree) distribution $\mathbf{n}$ (resp. $D$).
(2) Output: a rooted $m$-ary cacti having $\mathbf{n}$ as color distribution (resp. $D$ as degree distribution).
(3) uniform random generation of an $m$-cactus-like endofunction having $\mathbf{n}$ as color distribution (resp. $D$ as degree distribution):
    (a) compute a uniform random block partition with respect to

- either a color distribution $\mathbf{n}$: for $i = 1, \ldots, m$ generate a random binary word $B_i$ of length $b$ ($b-1$ if $i = 1$) with $n_i - 1$ occurrences of 1; deduce a block partition from the $B_i$'s and label the blocks (in any way) from 1 to $b$;
- or a degree distribution $D$: for $i = 1, \ldots, m$ generate a random permutation $P_i$ on the multiset $\{d_{i,2}, \ldots\}$; deduce a block partition from the $P_i$'s, add a random 1-block (the distinguished block) and label the blocks (in any way) from 1 to $b$;

(b) put $1_1$ in the preimage of 0,
(c) compute a uniform random word $W$ of length $m-1$ on $[p]$ such that the last letter is the label of a 1-block (resp. of the distinguished 1-block),
(d) compute the placement of minimum vertices other than $1_1$ associated to $W$,
(e) for every color, compute a random permutation on the non-minimum vertices of this color and put these vertices in the blocks according to these permutations;

(4) apply the transformation $\Psi^{-1}$ from $m$-cactus-like endofunction to $m$-cactus-like rooted trees described in subsection 2.4,
(5) apply the transformation from $m$-cactus-like rooted trees to rooted $m$-ary cacti,
(6) forget the labels on vertices to obtain an unlabelled rooted $m$-cacti.

The fact that these algorithms generate a rooted $m$-ary cactus having $\mathbf{n}$ as color distribution (resp. $D$ as degree distribution) follows immediately from the proofs of Theorems 3 and 4. Thus we have to check that these algorithms generate uniformly which follows directly from the fact that during the step 3 the block partitions and placements are uniformly generated and independent of each other.

**Remark 3.** It is possible to design data structures for rooted $m$-ary cacti that will allow to implement these algorithms in $O(n \log(m))$ worst-case time and $O(n)$ space (see [1]).

## 5. Conclusion

In this paper, we presented two simple bijective proofs of the arborescent form of the multivariable Good-Lagrange inversion formula and we illustrated their usefulness in the problems of enumeration and random generation of colored rooted trees and cacti. This method can also be applied to the exhaustive generation of such structures.

There is a related open problem: to design a proof of the arborescent form of the Good-Lagrange formula involving unlabelled structures and ordinary generating functions. Among all the proofs of the multivariable Good-Lagrange formula, as far as we know, the only "unlabelled proof" is due to Chottin [11] and is limited to the case of two variable series ($m = 2$). Such a proof would be of great interest, especially in the design of exhaustive generation algorithms of unlabelled colored combinatorial structures (like cacti for example). We can think to solve this problem by using the theory of $m$-colored linear species with ordinary generating functions. Indeed, in [23], Joyal gives, in the case of one variable, an unlabelled analog of the Labelle proof using linear species. It would be interesting to extend this proof to the multivariable case.

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