MACM 202 Assignment 5, Spring 2004

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Due in Tuesday March 23rd at 10:30am. Use Maple for all questions. Do each question in a separate Maple worksheet. Late penalty is 20% for each day late.

Question 1: Numerical Methods

(a) Consider the initial value problem $x'(t) = f(x(t))$ where $f(x) = x(1-x)$ and $x(0) = 0.1$. Compute the exact solution using `dsolve` and determine $x(3)$ numerically to 10 decimal places. The Taylor series method we saw in class was derived from

$$x(t_k + h) = x(t_k) + hx'(t_k) + \frac{h^2}{2}x''(t_k) + \frac{h^3}{6}x'''(t_k) + ...$$

using terms up to and including the term in h^2. If we use only terms up to order h we get Euler's method. Derive a new Taylor series method which uses terms up to order h^3. Thus we now have three "Taylor series methods". For each method, estimate $x(3)$ for $n = 1, 2, 4, ..., 256$ steps and compute the error e_n between the approximation x_n and exact answer, and for $n = 2, 4, ..., 256$ compute and display $n, x_n, e_{n/2}/e_n$. For each method, what rate of the convergence do you observe? Is the error proportional to h, h^2, h^3 or some other function of h?

(b) Apply Eulers method and Heun's method to $x'(t) = x(1-x)$ with $x(0) = 0.5$ to estimate $x(1)$ using $n = 4$ steps, and $n = 8$ steps. Using these two estimates for $x(1)$ estimate how many steps would be needed to reduce the error to 10^{-4}. Now run each method for the estimated number of steps and compare with the exact solution.

(c) Investigate what happens when h is too large in Euler's method when applied to solve $x'(t) = x(1-x)$ for $x(0) = 0.5$. For different values of h, for example $h = 0.2, 0.5, 1.0, 2.0,$ *etc.* run Euler's methods for $n = 50$ steps and plot the points

$$(t_0, x_0), (t_1, x_1), ..., (t_{50}, x_{50})$$

that you get and state what behaviour you see.

Question 2: Some DE Models

Do (c) and one of (a) or (b).

(a) Let P be a population that lives in a community. Suppose a virus spreads through the population and suppose everyone is susceptible to the virus, i.e., eventually everyone will become infected. Let $y(t)$ be the proportion of the population with the virus at time t. That is, $0 \leq y(t) \leq 1$ and the proportion of the population which does not have the virus yet is $1 - y(t)$. Model the spread of the virus through the population with a differential equation. For the model that you give, graph solution curves that illustrate the spread over time of the virus through the population for different initial values using the `DEplot` command.

(b) Let $x(t)$ denote a fish population at time t. Consider models of the form $x'(t) = G(x) - H(x,t)$ where $G(x)$ models the natural growth of the fish population and $H(x,t)$ represents a *seasonal harvest* taken by fishermen. Construct such a model where $H(x,t) = xg(t)$ where $0 \leq g(t) \leq 1$. Graphically display what happens to the fish population for appropriate choices of $x(0)$ using the `DEplot` command.

(c) Consider a lake of constant volume V. Let $Q(t)$ be the amount of pollutant in the lake at time t and let $c(t) = Q(t)/V$ be the concentration of pollutant at time t. Suppose pollutant enters the lake through rivers entering the lake and directly from factories bordering the lake. Suppose pollutant exits the lake via the outlet river. Assume also that the pollutant is always evenly distributed throughout the lake. The problem is to determine $c(t)$. Let the concentration of pollutant entering the lake by rivers at rate r be k. Let the rate at which water leaves the lake also be r. And suppose factories add pollutant to the lake at a constant rate P.

   (i) Find an expression for $c'(t)$. Let $c(0) = c_0$. Solve the differential equation for $c(t)$. You should obtain

$$c(t) = k + \frac{P}{r} - e^{-\frac{r}{V}t}\,(kr + P - c_0\,r)\,r^{-1}.$$

   (ii) If the addition of pollutants to the lake is terminated at some time t, i.e., $k \to 0$ and $P \to 0$, determine the time T that must elapse before the concentration $c(t)$ of pollutant is reduced to 10% of its original value.

   (iii) Using the given data, determine from part (b) the time T necessary to reduce the pollutant to 10% of the original value.

Lake	V (km$^3 \times 10^{+3}$)	r (km^3/year)
Superior	12.2	65.3
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

Question 3: Second order ODEs

(a) Do problem 5.2 in the text. Give one set of values for L, R, C such that the solution is NOT oscillatory and one set for which is IS oscillatory. Verify your choice by solving the DE using `dsolve`. For initial values use $q'(0) = 0$ and $q(0) = 1$. To specify $q'(0) = 0$ in Maple use $D(q)(0) = 0$.

(b) Consider $y'' = -4y' - ky$ with $y(0) = 1, y'(0) = 0$. Solve the DE using the characteristic equation for $k = 3, 4, 8$. For the solution which is oscillatory, express your answer in the form $y(t) = e^{kt}[c_1 \sin(\omega t) + c_2 \cos(\omega t)]$ and in the form $y(t) = Ae^{kt} \sin(\omega t + \phi)$.