Lec20C Introduction to Risch Integration Algorithm

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11.2 Basic Concepts of Differential Algebra

Theorem 11.1. Let F be a differential field and $f, g \in F$.

(i) $D(0) = 0$ and $D(1) = 0$.

(ii) $D(-f) = -D(f)$

(iii) $D(f/g) = [D(f)g - f \cdot D(g)]/g^2$

(iv) $D(f^n) = n \cdot D(f) \cdot f^{n-1}$ for $n \in \mathbb{N}$.

Proof (i) $\underset{-D(0)}{\underline{D(0)}} = D(0+0) \overset{P1}{=} \underset{-D(0)}{\underline{D(0)}} + \underline{D(0)} \Rightarrow D(0) = 0$.

(iii) $D(f/g) = D(f \cdot g^{-1}) \overset{P2}{=} \underline{D(f) \cdot g^{-1}} + f \cdot D(g^{-1}) = \dfrac{D(f)}{g} - \dfrac{f D(g)}{g^2}$.

$0 = D(1) = D(g \cdot g^{-1}) = D(g) g^{-1} + g \cdot D(g^{-1}) \Rightarrow D(g^{-1}) = -\dfrac{D(g)}{g} \cdot g^{-1}$

Is $\mathbb{Q}(x)$ a differential field?

$\mathbb{Q}(x) = \{ \frac{a}{b} : a, b \in \mathbb{Q}[x], b \neq 0 \}$ $\quad \frac{d}{dx} \frac{a}{b} = \dfrac{a'b - ab'}{b^2} \in \mathbb{Q}(x)$. ✓

Is $F = \mathbb{Q}[\ln x] = \{ \sum_{i=0}^{n} a_i (\ln x)^i : a_i \in \mathbb{Q} \}$.

$1 \cdot \ln x \in F$ $\quad \frac{d}{dx} \ln x = \frac{1}{x} \notin F$

So F is not a differential ring w.r.t. $\frac{d}{dx}$

Take $F = \mathbb{Q}(x)[\ln(x)] = \{ \sum_{i=0}^{n} a_i(x) (\ln x)^i : a_i(x) \in \mathbb{Q}(x) \}$

$\left(\frac{x}{1-x}\right) \cdot \ln x \in F$.

$\frac{d}{dx} \underline{a_1(x) \cdot \ln x} = \underset{\overset{\cap}{\mathbb{Q}(x)}}{a'(x)} \cdot \ln x + \underset{\overset{\cap}{\mathbb{Q}(x)}}{\underbrace{\underline{a_1(x)} \cdot \frac{1}{x}}} \in \mathbb{Q}(x)[\ln x]$

The Risch Integration Algorithm Ch 12 (Bob Risch 1968)

$f(x) \in E$ $\longrightarrow$ $\boxed{\begin{array}{l} \text{Risch} \\ \text{Int.} \\ \text{Alg. Ch12} \end{array}}$ $\longrightarrow$ $g \in E$ s.t. $g' = f$

$\ln x$ $\qquad$ $x\ln x - x$

OR

FAIL $\nexists\, g \in E$ s.t. $g' = f$

E.g. $\int e^{-x^2} dx \longrightarrow$ FAIL.

Trap. $\int \left(\frac{e^x}{x} + e^x \ln x\right) dx = \int \frac{e^x}{x} dx + \int e^x \ln x \, dx =$ FAIL.

$\int \frac{e^x}{x} dx \xrightarrow{\text{RISCH}}$ FAIL $\qquad \int e^x \ln x\, dx \xrightarrow{\text{RISCH}}$ FAIL

This is elementary

$= \ln x \cdot e^x + c.$

Consider $\int e^{-x} (\underbrace{\ln 4 - 2\ln 2}_{=0}) \, dx = \int 0 \, dx = C.$

Assumption: now how to test for zero in the constant field $\mathbb{C}$.

Risch Structure Theorem.

Consider $\int e^{-x^2} [\underbrace{\sin 2x - 2\sin x \cos x}_{=0}] \, dx = C.$

Consider $\int \frac{e^x f(x)}{1 - e^{x/2} + e^{2x}} dx = \int \frac{\theta_1^2}{1 - \theta_1 + \theta_1^4} dx$ where $\theta_1 = e^{\frac{x}{2}}$

$\mathbb{Q}(x)(\theta_1 = e^x, \theta_2 = e^{x/2}, \theta_3 = e^{2x})$

$\theta_3 = \theta_1^2 \quad \theta_1 = \theta_2^2$

$f(x) \in \mathbb{Q}(x)(\theta_1 = e^{x/2})$