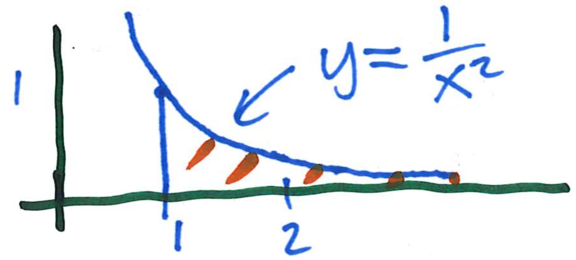
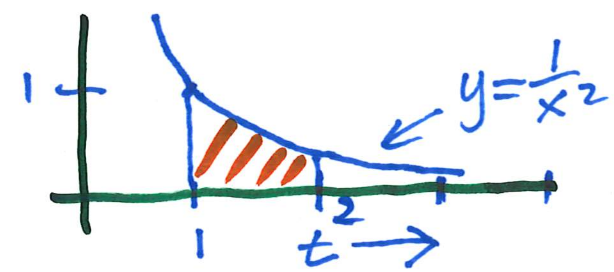
7.8 Improper Integrals

Reading Week break next week.

Consider $\int_1^\infty \frac{1}{x^2}\,dx$





$$\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = \left(-\frac{1}{t}\right)-(-1) = 1 - 1/t$$

$$\frac{d}{dx}\left(-x^{-1}\right) = \quad +x^{-2}$$

$t=2 \quad 1-1/2 = \frac{1}{2}$

$t=4 \quad 1-1/4 = 3/4$

$t=8 \quad 1-1/8 = 7/8$

$\downarrow \qquad \downarrow$

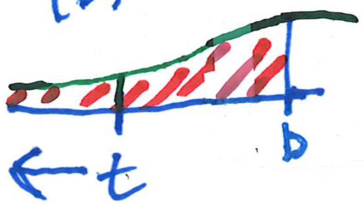
$1 \;=\; 1$

Idea: $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(1-\underbrace{\frac{1}{t}}_{\to 0}\right) = 1.$$

Improper Integrals of type I

(a) If $\int_a^t f(x)\,dx$ exists for $t \geq a$ then $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$
if the limit exists.

(b)

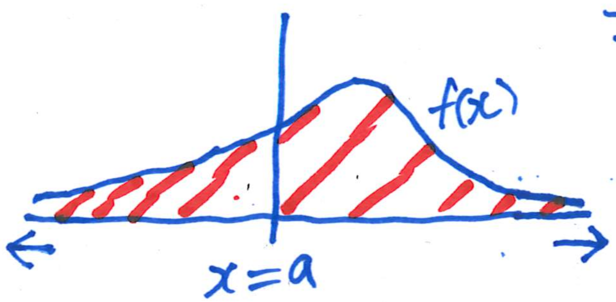


If $\int_t^b f(x)\,dx$ exists for $t \leq b$ then $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to -\infty} \int_t^b f(x)\,dx$
if the limit exists.

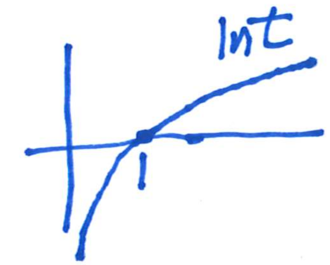
The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called
__convergent__ if the limit exists and __divergent__ otherwise.

(c)

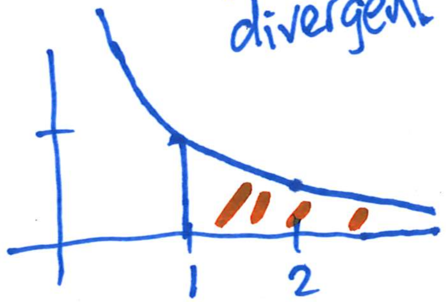
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$ if __both__ integrals are convergent.



Ex 1 $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\Big[\ln(x)\Big]_1^t = \lim_{t\to\infty}\big(\underset{\substack{\downarrow\\ \infty}}{\ln(t)} - \underset{\substack{\shortparallel\\ 0}}{\ln 1}\big) = \infty$

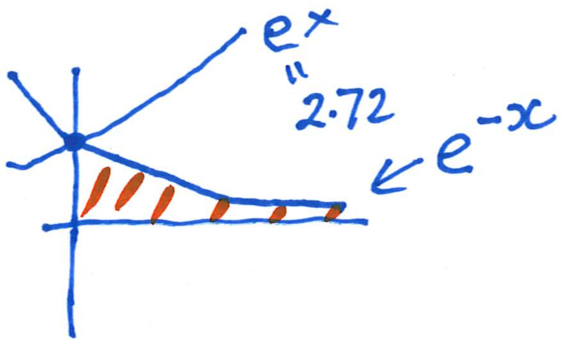


divergent

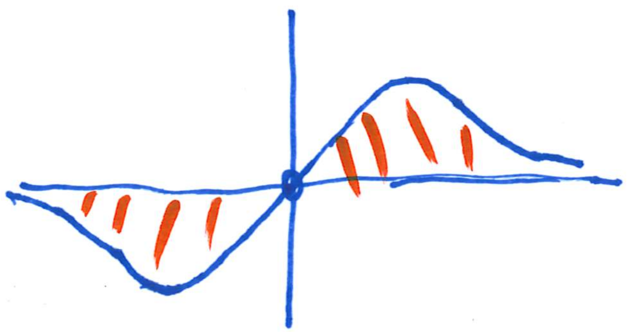


Ex 2 $\displaystyle\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\Big[-e^{-x}\Big]_0^t = \lim_{t\to\infty}\big(-e^{-t} - -e^{0}\big)$

$\displaystyle = \lim_{t\to\infty}\big(\underset{\substack{\downarrow\\ 0}}{-e^{-t}} + \underset{\substack{\shortparallel\\ 1}}{1}\big) = 1.$



Ex 3 $\displaystyle\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx = -\frac{1}{2} + \frac{1}{2} = \underline{0}$



$\displaystyle\int_{-\infty}^{0} xe^{-x^2}dx = \lim_{t\to-\infty}\int_{t}^{0} xe^{-x^2}dx$ .... exercise .... $= \underline{\underline{-\frac{1}{2}}}$

$$\int_0^0 xe^{-x^2}dx = \int \frac{xe^{-u}du}{2x} = \int \frac{1}{2}e^{-u}du = -\frac{1}{2}e^{-u} + c = -\frac{1}{2}e^{-x^2} + c.$$

Let $u = x^2$

$du/dx = 2x \Rightarrow dx = \frac{du}{2x}$

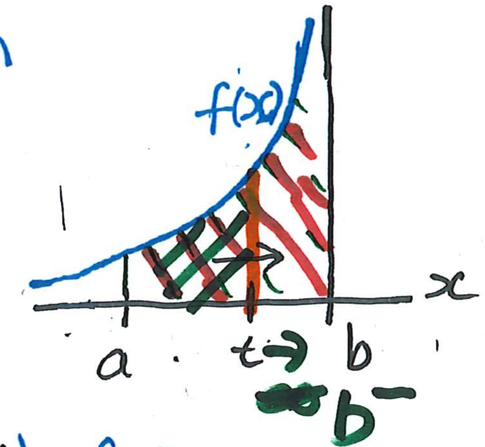
$$\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}\right) - \left(-\frac{1}{2}\cdot e^0\right) = \frac{1}{2}$$

$$= \lim_{t\to\infty}\left(\underbrace{\frac{1}{2}}_{= 1/2} - \underbrace{\frac{1}{2}e^{-t^2}}_{\to 0}\right)$$

Improper Integrals of type II

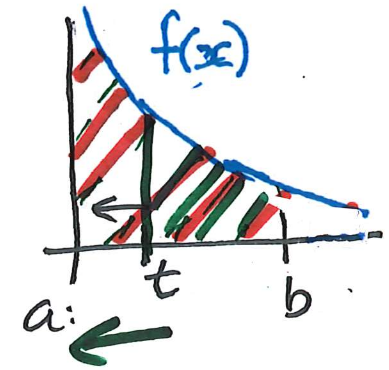
(a) If f is continuous on $[a,b)$ and discontinuous at b then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx \quad \text{provided the limit exists.}$$



(b) If f is continuous on $(a,b]$ and discontinuous at a then

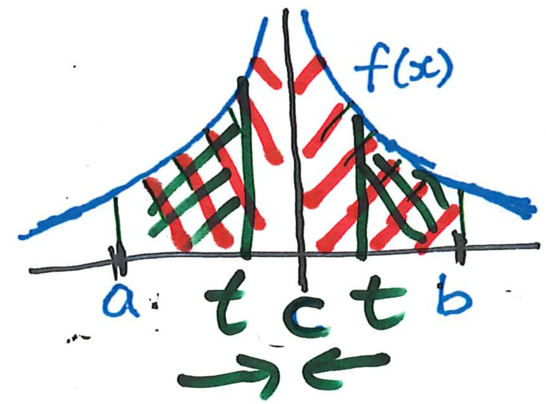
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \quad \text{provided the limit exists.}$$



These improper integrals are called <u>convergent</u> if the limits exist and <u>divergent</u> if not

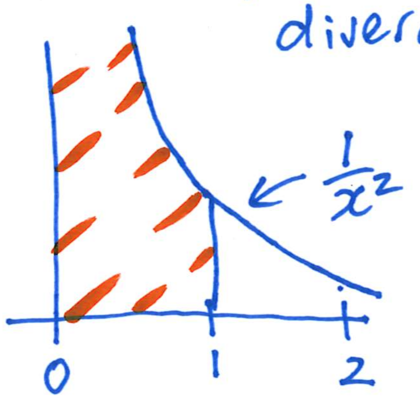
(c) If f has a discontinuity at c where $a<c<b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent then we define

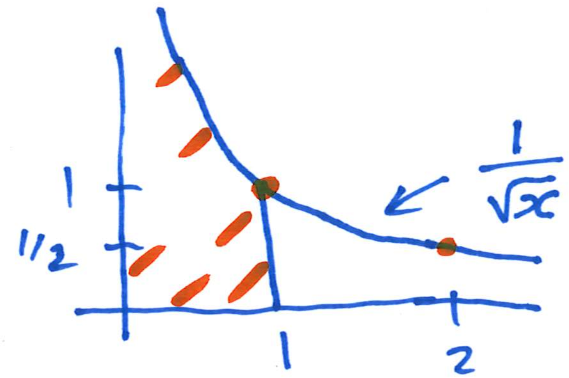
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



① $\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\left[-\frac{1}{x}\right]_t^1 = \lim_{t\to 0^+}\Big(-1+\underbrace{\frac{1}{t}}_{\downarrow\ +\infty}\Big) = \infty$

↑ divergent

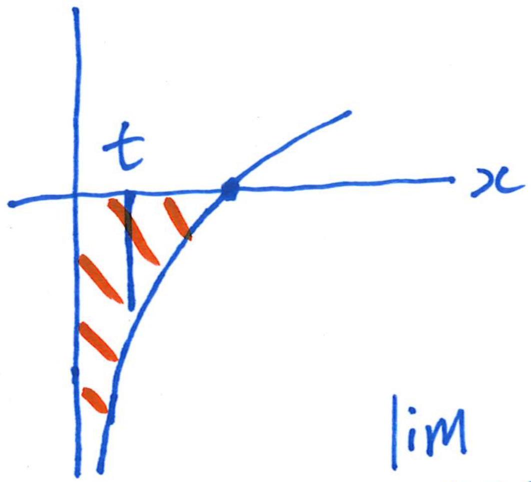




② $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^1 = \underline{\lim_{t\to 0^+}}\Big(2-2\underbrace{\sqrt{t}}_{\downarrow\ 0}\Big) = 2.$

$\int x^{-1/2}\,dx = 2x^{1/2}+c = 2\sqrt{x}+c$

③ $$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx = \lim_{t\to 0^+}\Big[x\ln x - x\Big]_t^1$$



$$\int \underset{\substack{\uparrow\\ g'}}{1}\cdot\underset{\substack{\uparrow\\ f}}{\ln x} = \lim_{t\to 0^+}\Big(\big(1\cdot\underset{\substack{\shortparallel\\ -1}}{\ln 1 - 1}\big) - \big(\underbrace{\underset{\substack{\downarrow\\ 0}}{t}\,\underset{\substack{\downarrow\\ -\infty}}{\ln t}}_{\to 0} - \underset{\substack{\downarrow\\ 0}}{t}\big)\Big) = -1.$$

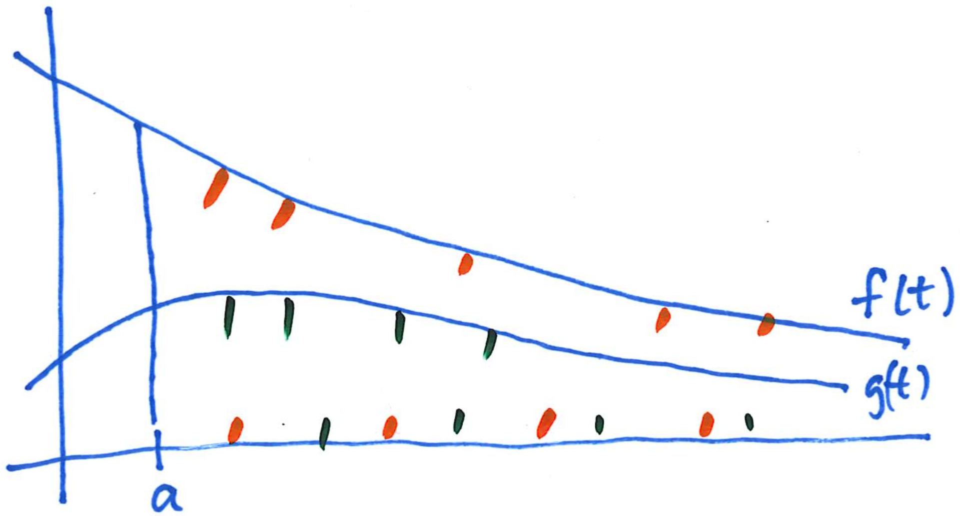
$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t \to -\infty}{\underset{\substack{\shortparallel\\ t^{-1}}}{1/t} \to +\infty} = \lim_{t\to 0^+}\left(\frac{1/t}{-1/t^2} = -t\right) = 0$$

<u>L'Hôpital's rule</u>

$$\lim_{t\to 0^+}\frac{f(t)\to\pm\infty}{g(t)\to\pm\infty} = \lim_{t\to 0^+}\frac{f'(t)}{g'(t)}$$

Comparison Theorem

Suppose $f(t) \geq g(t) \geq 0$ on $[a, \infty]$



① If $\int_a^\infty f(t)\,dt$ is convergent then $\int_a^\infty g(t)\,dt$ is convergent.

② If $\int_a^\infty g(t)\,dt$ is divergent (infinite) then $\int_a^\infty f(t)\,dt$ is divergent.