LINEARIZATION OF MATRIX POLYNOMIALS EXPRESSED IN POLYNOMIAL BASES

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Abstract

Companion matrices of matrix polynomials $L(\lambda)$ (with possibly singular leading coefficient) are a familiar tool in matrix theory and numerical practice leading to so-called "linearizations" $\lambda B - A$ of the polynomials. Matrix polynomials as approximations to more general matrix functions lead to the study of matrix polynomials represented in a variety of classical systems of polynomials, including orthogonal systems and Lagrange polynomials, for example. For several such representations, it is shown how to construct (strong) linearizations via analogous companion matrix pencils. In case $L(\lambda)$ has Hermitian or alternatively complex symmetric coefficients, the determination of linearizations $\lambda B - A$ with A and B Hermitian or complex symmetric is also discussed.

1 Introduction

An $s \times s$ matrix polynomial $P(\lambda)$ of degree n has s^2 entries, each of which is a scalar (complex) polynomial in λ with degree not exceeding n. Grouping like powers of λ together determines the representation $P(\lambda) = \sum_{j=0}^{n} \lambda^j A_j$, where the coefficients $A_j \in \mathbb{C}^{s\times s}$ and $A_n \neq 0$. Clearly, the polynomial could also be uniquely determined by $n+1$ samples of the function: $P_j := P(z_j)$, where the points $z_0, z_1, \ldots, z_n \in \mathbb{C}$ are distinct.

The process of gathering the $n+1$ matrices of coefficients of the successive powers of λ could be described as "interpolation by monomials". Indeed, the matrices $P_0, P_1, \ldots, P_n$ may be samples of a function $\hat{P}(\lambda)$ of a more general type; analytic, for example, and one may be interested in how the interpolant $P(\lambda)$ approximates $\hat{P}(\lambda)$.

We consider only matrix polynomials which are *regular* in the sense that the determinant, $\det P(\lambda)$, does not vanish identically. Practical and algorithmic concerns with

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such polynomials frequently involve the determination of eigenvalues; namely, those $\lambda_0 \in \mathbb{C}$ for which the rank of $P(\lambda_0)$ is less than s. Thus, the eigenvalue multiplicity properties (geometric and algebraic) have a role to play.

It is natural to study spectral properties of the polynomial via the associated pencil $\lambda C_1 - C_0$, where (when $n = 4$, for example)

$$C_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 0 & 0 & -A_0 \\ I & 0 & 0 & -A_1 \\ 0 & I & 0 & -A_2 \\ 0 & 0 & I & -A_3 \end{bmatrix}. \tag{1}$$

This has been extensively used and recognised; see [12] and [13], for example, among many other sources. The vital property of this pencil is that it forms a "strong" linearization of $P(\lambda)$ in the sense that it reproduces the multiplicity structures of the eigenvalues of $P(\lambda)$, both finite and infinite. An infinite eigenvalue is said to exist when $\det A_n = 0$ (see [12] and [18], in particular).

A major objective of this paper is to consider the analogous problems which arise when $P(\lambda)$ is represented in other bases (i.e. other than monomials) for the linear space of scalar polynomials with degree not exceeding n. Applications of matrix polynomials in other bases occur in Computer-Aided Geometric Design (where Bernstein bases are used) and in the Lagrange basis (see e.g. [4]). In this present paper, analogues of (1) are to be formulated, and the property of strong linearization is to be investigated, i.e. linearizations which preserve the invariant polynomials of *both* $P(\lambda)$ and its *reverse* $P^{\sharp}(\lambda) := \lambda^n P(1/\lambda)$.

The details of this program depend on a particular property of the polynomial basis employed: whether it is *degree-graded* (consists of polynomials of degrees $0, 1, 2, \ldots, n$ (like the monomials)), or whether all polynomials have the same degree (as with the Lagrange interpolating polynomials). The paper is organised accordingly: Sections 2 and 3 are concerned with degree-graded bases. Sections 4 and 5 discuss interpolation with Bernstein and Lagrange bases, respectively.

It will be seen that the strategy adopted below (as in [1]) involves the determination of λ-dependent (triangular) LU-decompositions of $\lambda C_1 - C_0$ (and its various analogues). We remark that in some algorithms (especially of Rayleigh-quotient type) it is necessary to solve linear systems $(\lambda_0 C_1 - C_0)x = b$ (with a fixed λ_0) many times. The LU decompositions used here can also play a useful role in this algorithmic context. There is an exhaustive study of these LU factorization in [3].

Bases other than the monomials find many applications. For problems in computer-aided geometric design, the Bernstein-Bézier basis and the Lagrange basis are most useful (see [10], for example). There are problems in partial differential equations with symmetries in the boundary conditions where Legendre polynomials are the most natural. Finally, in approximation theory, Chebyshev polynomials have a special place due to their minimum-norm property (see e.g. [22]).

2 Degree-graded polynomial bases

2.1 Linearization

Real polynomials $\{\phi_n(\lambda)\}_{n=0}^{\infty}$ with $\phi_n(\lambda)$ of degree n which are orthonormal on an interval of the real line (with respect to some nonnegative weight function) necessarily satisfy a three-term recurrence relation (see Chapter 10 of [8], for example). These relations can be written in the form

$$\lambda\phi_j(\lambda) = \alpha_j\phi_{j+1}(\lambda) + \beta_j\phi_j(\lambda) + \gamma_j\phi_{j-1}(\lambda), \qquad j = 1, 2, \ldots, \tag{2}$$

where the $\alpha_j,\ \beta_j,\ \gamma_j$ are real, $\phi_{-1}(\lambda) = 0$, $\phi_0(\lambda) = 1$, and, if k_j is the leading coefficient of $\phi_j(\lambda)$,

$$0 \neq \alpha_j = \frac{k_j}{k_{j+1}}, \qquad j = 0, 1, 2, \ldots. \tag{3}$$

The choices of coefficients $\alpha_j,\ \beta_j,\ \gamma_j$ defining three well-known sets of orthogonal polynomials (asociated with the names of Chebyshev and Legendre) are summarised in Table 1. Such orthogonal polynomials have well-established significance in mathematical physics and numerical analysis (see e.g. [11]). More generally, any sequence of polynomials $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ with $\phi_j(\lambda)$ of degree j is said to be *degree-graded* and obviously forms a linearly independent set; but is not necessarily orthogonal.

Table 1: Three well-known orthogonal polynomials

Polynomial	$T_n(x)$	$P_n(x)$	$C_n(x)$
Name of polynomial	Chebyshev(1st kind)	Legendre(Spherical)	Chebyshev(2nd kind)
Weight function	$(1-x^2)^{-\frac{1}{2}}$	1	$(1-x^2)^{-\frac{1}{2}}$
Orthogonality interval	$[-1, 1]$	$[-1, 1]$	$[-1, 1]$
Leading coefficient k_n	2^{n-1}	$\frac{(2n)!}{2^n(n!)^2}$	2^n
α_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{n+1}{2n+1}$
β_n	0	0	0
γ_n	1	1	$\frac{n}{2n+1}$

An $s \times s$ matrix polynomial $P(\lambda)$ of degree n can now be written in terms of a set of degree-graded polynomials:

$$P(\lambda) = A_n\phi_n(\lambda) + A_{n-1}\phi_{n-1}(\lambda) + \cdots + A_1\phi_1(\lambda) + A_0\phi_0(\lambda). \tag{4}$$

For convenience, let us assume $n = 5$ and the generalizations for all positive n will be clear. Define block-matrices

$$C_0 = \begin{bmatrix} \beta_0 I_s & \gamma_1 I_s & 0 & 0 & -k_4A_0 \\ \alpha_0 I_s & \beta_1 I_s & \gamma_2 I_s & 0 & -k_4A_1 \\ 0 & \alpha_1 I_s & \beta_2 I_s & \gamma_3 I_s & -k_4A_2 \\ 0 & 0 & \alpha_2 I_s & \beta_3 I_s & -k_4A_3 + k_5\gamma_4A_5 \\ 0 & 0 & 0 & \alpha_3 I_s & -k_4A_4 + k_5\beta_4A_5 \end{bmatrix}, \tag{5}$$

$$C_1 = \begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 & 0 \\ 0 & 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 & k_5A_5 \end{bmatrix}, \tag{6}$$

(and observe how the matrices of (1) fit into this scheme). This construction is essentially that of a "comrade" matrix introduced by Barnett; see Chapter 5 of [5] and [6].

A little computation shows that

$$\left[\begin{array}{ccccc} \phi_0(\lambda)I_s & \phi_1(\lambda)I_s & \phi_2(\lambda)I_s & \phi_3(\lambda)I_s & \phi_4(\lambda)I_s \end{array}\right](\lambda C_1 - C_0) = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & k_4 P(\lambda) \end{array}\right]. \quad (7)$$

The first $n-1$ row-into-column products simply reproduce some of the relations (2). For the last such product use equations (2), (3), and (4). In the more suggestive notation of [21] this equation reads: [2]

$$\left(\mathbf{\Phi}^T(\lambda) \otimes I\right)(\lambda C_1 - C_0) = k_{n-1} e_n^T \otimes \mathbf{P}(\lambda)$$

where $\mathbf{\Phi}^T(\lambda) = [\phi_0(\lambda), \phi_1(\lambda), \ldots, \phi_{n-1}(\lambda)]$.

Now suppose that λ_0 is an eigenvalue of $P(\lambda)$ with left eigenvector y, i.e. $y^H P(\lambda_0) = 0$ (where the superscript $()^H$ denotes the Hermitian (complex-conjugate) transpose of a matrix or vector). Then evaluating (7) at λ_0 and premultiplying by y^H gives:

$$\left[\begin{array}{ccccc} \phi_0(\lambda_0)y^H & \phi_1(\lambda_0)y^H & \phi_2(\lambda_0)y^H & \phi_3(\lambda_0)y^H & \phi_4(\lambda_0)y^H \end{array}\right](\lambda_0 C_1 - C_0) = 0\,. \quad (8)$$

This shows that every finite eigenvalue of $P(\lambda)$ is also an eigenvalue of $\lambda C_1 - C_0$ and also shows how left eigenvectors of $\lambda C_1 - C_0$ can be generated from those of $P(\lambda)$. (This is a generalization of part(ii) of Theorem 5.2 of [5]; special cases have appeared in [1].) The *l*eft eigenvectors do not have special role in this discussion. A similar explicit characterization of the relationship of a right eigenvector w of $P(\lambda)$ corresponding to finite eigenvalue λ with a right eigenvector of the pencil $\lambda C_1 - C_0$ can be made (see [1]).

This argument shows that $P(\lambda)$ and $\lambda C_1 - C_0$ have the same spectrum, but more is true. To establish this a Lemma on linearizations is required. A linearization of the regular matrix polynomial $P(\lambda)$ is generally defined to be an $sn \times sn$ pencil $\lambda A - B$ for which

$$E(\lambda)(\lambda A - B)F(\lambda) = \left[\begin{array}{cc} I_{n(s-1)} & 0 \\ 0 & P(\lambda) \end{array}\right], \quad (9)$$

for some unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$. We need a more general characterization of a linearization as follows:

Lemma 1 *If* (9) *holds for functions $E(\lambda)$ and $F(\lambda)$ which are unimodular and analytic on a neighbourhood of the spectrum of $P(\lambda)$, then $\lambda A - B$ is a linearization of $P(\lambda)$.*

Proof: A linearization $\lambda A - B$ can be characterized by the property that all of its eigenvalues and their partial multiplicities (including the eigenvalue at infinity if A_n is singular) are the same as those of $P(\lambda)$ (Theorem A.6.2 of [13], for example). The fact that these properties are preserved by the more general matrix functions, $E(\lambda)$ and $F(\lambda)$, follows immediately from Theorem A.6.6 of [13]. □

Remark. Let Z_ϕ be the set of all zeros of $\phi_1(\lambda), \cdots, \phi_{n-1}(\lambda)$. This set is necessarily finite. We will see that to use the above lemma correctly, we will have to *block pivot* whenever λ is in a small enough neighbourhood of any of these zeros. It follows from the work of [1, 3] that this can always be done.

[2] The authors thank a helpful reviewer for pointing out the connections with work of [16] and [21]. This equation shows a clear connection with the 'left ansatz' of [21, eq. (3.9)]. This analogy suggests that, as in [21], for each polynomial basis $\mathbf{\Phi}(\lambda)$ two vector spaces of linearizations may be defined, and that, as in [16], these vector spaces may be explored for linearizations that preserve structure, or are particularly well-suited for the task at hand. These considerations deserve further study.

Theorem 2 *Let $P(\lambda)$ be a matrix polynomial of degree n and $\{\phi_n(\lambda)\}_{n=0}^{\infty}$ be a degree-graded system of polynomials satisfying the recurrence relation* (2)*. Then the pencil $\lambda C_1 - C_0$ defined by* (5) *and* (6) *is a strong linearization of $P(\lambda)$.*

Proof: First, assume that Z_ϕ, the set of all zeros of $\phi_1(\lambda), \cdots, \phi_{n-1}(\lambda)$ does not intersect the set of all eigenvalues of $P(\lambda)$. In [3] the λ-dependent block LU factors of $\lambda C_1 - C_0$ for a pencil of the form (5)– (6) and of degree n are explicitly given as follows:

$$L(\lambda) = \begin{bmatrix} I_s & & & \\ -\frac{\phi_0(\lambda)}{\phi_1(\lambda)} I_s & I_s & & \\ & \ddots & \ddots & \\ & & -\frac{\phi_{n-2}(\lambda)}{\phi_{n-1}(\lambda)} I_s & I_s \end{bmatrix}, \tag{10}$$

$$U(\lambda) = \begin{bmatrix} \alpha_0 \frac{\phi_1(\lambda)}{\phi_0(\lambda)} I_s & -\gamma_1 I_s & & & U_{1,n}(\lambda) \\ & \ddots & \ddots & & \vdots \\ & & \alpha_{n-3} \frac{\phi_{n-2}(\lambda)}{\phi_{n-3}(\lambda)} I_s & -\gamma_{n-2} I_s & U_{n-2,n}(\lambda) \\ & & & \alpha_{n-2} \frac{\phi_{n-1}(\lambda)}{\phi_{n-2}(\lambda)} I_s & U_{n-1,n}(\lambda) \\ & & & & U_{n,n}(\lambda) \end{bmatrix}, \tag{11}$$

where

$$U_{i,n}(\lambda) = \begin{cases} k_{n-1} A_0, & i = 1 \\ k_{n-1} A_{j-1} + \frac{\phi_{j-2}(\lambda)}{\phi_{j-1}(\lambda)} U_{j-1,n}(\lambda), & i = 2{:}(n-2) \\ k_{n-1} A_{n-2} + \frac{\phi_{n-3}(\lambda)}{\phi_{n-2}(\lambda)} U_{n-2,n}(\lambda) - k_n \gamma_{n-1} A_n, & i = n-1 \\ \frac{\phi_0(\lambda)}{(\alpha_0 \cdots \alpha_{n-2}) \phi_{n-1}(\lambda)} P(\lambda). & i = n \end{cases} \tag{12}$$

Clearly, $L(\lambda)$ is nonsingular for all $\lambda \notin Z_\phi$. For such λ, $\det(L(\lambda)) \equiv 1$. Thus, $U(\lambda)$ is singular at the eigenvalues of $P(\lambda)$. If we define $\widetilde{U}(\lambda)$ to be the same as $U(\lambda)$ except for its last block entry which is replaced by

$$\widetilde{U}_{n,n}(\lambda) = \frac{\phi_0(\lambda)}{(\alpha_0 \cdots \alpha_{n-2}) \phi_{n-1}(\lambda)} I_s, \tag{13}$$

then $\widetilde{U}(\lambda)$ is also nonsingular and $\det(\widetilde{U}(\lambda)) \equiv 1$. Now, we can construct the unimodular matrices $E(\lambda) = L^{-1}$ and $F(\lambda) = \widetilde{U}^{-1}$ as follows:

$$E(\lambda) = \begin{bmatrix} I_s & & & & \\ \frac{\phi_0(\lambda)}{\phi_1(\lambda)} I_s & I_s & & & \\ \frac{\phi_0(\lambda)}{\phi_2(\lambda)} I_s & \frac{\phi_1(\lambda)}{\phi_2(\lambda)} I_s & I_s & & \\ \vdots & \vdots & & \ddots & \\ \frac{\phi_0(\lambda)}{\phi_{n-1}(\lambda)} I_s & \frac{\phi_1(\lambda)}{\phi_{n-1}(\lambda)} I_s & \cdots & \frac{\phi_{n-2}(\lambda)}{\phi_{n-1}(\lambda)} I_s & I_s \end{bmatrix}, \tag{14}$$

$$F_{i,j}(\lambda) = \begin{cases} \frac{\phi_{i-1}(\lambda)}{\alpha_{i-1}\phi_i(\lambda)} I_s, & i = j = 1{:}(n-1) \\ \frac{\phi_{n-1}(\lambda)}{\phi_0(\lambda)} (\alpha_0 \cdots \alpha_{n-2}) I_s, & i = j = n \\ \frac{\gamma_{j-1}\phi_{j-1}(\lambda)}{\alpha_{j-1}\phi_j(\lambda)} F_{i,j-1}(\lambda), & i = 1{:}(n-2); j = (i+1){:}(n-1) \\ -\frac{\alpha_0 \cdots \alpha_{n-2}}{\phi_0(\lambda)} (k_{n-1} \sum_{k=0}^{n-2} A_k \phi_k(\lambda) - k_n \gamma_{n-1} A_n \phi_{n-2}(\lambda)), & i = n-1; j = n \\ -\frac{\alpha_0 \cdots \alpha_{n-2}}{\alpha_{i-1}} \frac{k_{n-1}\phi_{n-1}(\lambda)}{\phi_0(\lambda)\phi_i(\lambda)} \sum_{k=0}^{i-1} A_k \phi_k(\lambda) + \widehat{F}_{i,n}(\lambda), & i = (n-2){:}1; j = n \end{cases} \tag{15}$$

where

$$\widehat{F}_{i,n}(\lambda) = \begin{cases} F_{n-1,n}(\lambda), & i = n-1 \\ \frac{\gamma_i \phi_{i-1}(\lambda)}{\alpha_i \phi_i(\lambda)} \widehat{F}_{i+1,n}(\lambda), & i = (n-2){:}1 \end{cases} \tag{16}$$

Now a straightforward computation shows that:

$$\begin{bmatrix} I_{(n-1)s} & 0 \\ 0 & P(\lambda) \end{bmatrix} = E(\lambda)(\lambda C_1 - C_0) F(\lambda) \tag{17}$$

and, using Lemma 1, this shows that $\lambda C_1 - C_0$ is a linearization of $P(\lambda)$.

To show that this linearization is strong, we must show that unimodular matrices $H(\lambda)$ and $K(\lambda)$ exist such that:

$$\begin{bmatrix} I_{(n-1)s} & 0 \\ 0 & P^\sharp(\lambda) \end{bmatrix} = H(\lambda)(C_1 - \lambda C_0) K(\lambda) \tag{18}$$

where $P^\sharp(\lambda) = \lambda^n P(\frac{1}{\lambda})$. In fact, considering the LU factors of $\lambda C_1 - C_0$, we can write the LU factors of the reverse pencil $C_1 - \lambda C_0$ as follows:

$$L_1(\lambda) = L(\frac{1}{\lambda}) \quad \text{and} \quad U_1(\lambda) = \lambda U(\frac{1}{\lambda}). \tag{19}$$

Now we can let $H(\lambda) = L_1^{-1}(\lambda) = L^{-1}(\frac{1}{\lambda}) = E(\frac{1}{\lambda})$ where $E(\lambda)$ is given in (14), and $K(\lambda) = \widetilde{U}_1^{-1}$ where $\widetilde{U}_1$ is the same as U_1 except for the very last block entry which is replaced by:

$$\widetilde{U_1}_{n,n}(\lambda) = \frac{\phi_0(\frac{1}{\lambda})}{(\alpha_0 \cdots \alpha_{n-2})\lambda^{n-1}\phi_{n-1}(\frac{1}{\lambda})} I_s. \tag{20}$$

Using (15), we can construct $K(\lambda)$ as follows:

$$K_{i,j}(\lambda) = \begin{cases} \frac{F_{i.j}(\frac{1}{\lambda})}{\lambda}, & i = 1{:}(n-1); j = i{:}(n-1) \\ \lambda^{n-1} F_{i,n}(\frac{1}{\lambda}), & i = 1{:}n; j = n \end{cases} \tag{21}$$

and now it can be verified that (18) holds.

If instead $\lambda^* \in Z_\phi$, then, from the results of [1, 3], there exists a unimodular block pivot matrix Π, a neighbourhood B_ε of λ^*, and matrices $E(\lambda)$ and $F(\lambda)$ analytic in $B_\varepsilon(\lambda^*)$ such that all the factorings above may be computed *mutatis mutandis*. □

2.2 Symmetrizing the linearization

If the data matrices A_0, A_1, $\ldots$, A_n are Hermitian, then the resulting polynomial $P(\lambda)$ is Hermitian for real λ. Although the symmetry appears to be lost in the pencil $\lambda C_1 - C_0$, it can be recovered in the monomial case (when A_n is nonsingular) on postmultiplication of the companion matrix

$$C_0 C_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & -A_0 A_4^{-1} \\ I & 0 & 0 & -A_1 A_4^{-1} \\ 0 & I & 0 & -A_2 A_4^{-1} \\ 0 & 0 & I & -A_3 A_4^{-1} \end{bmatrix}.$$

by the Hermitian "symmetrizer",

$$H_0 := \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & 0 \\ A_3 & A_4 & 0 & 0 \\ A_4 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

In this way the eigenvalue problem for the Hermitian matrix polynomial $P(\lambda)$ can be examined in terms of the Hermitian pencil $\lambda H_0 - (C_0 C_1^{-1}) H_0$. This also works if the data matrices are not Hermitian but rather complex symmetric $A_j^T = A_j$. Such matrices occur in practice, for example with the symmetric Bezout matrix of a pair of bivariate polynomials with complex coefficients. In either case, the block symmetries of such a pencil can provide computational advantages and, as well, there is an extensive theory for problems of this kind developed in [13].

It turns out that, in some cases, this symmetrizing property extends to the pencils generated by other bases. Indeed, the following proposition is easily verified:

Proposition 3 *Let $\{\phi_n(\lambda)\}_{n=0}^{\infty}$ be a degree-graded system of polynomials satisfying a recurrence relation* (2) *in which $\alpha_j = \alpha \neq 0$, $\beta_j = \beta$, and $\gamma_j = \gamma$ for all j. Moreover, let $P(\lambda)$ be a Hermitian matrix polynomial defined in that basis with A_n nonsingular. Then, when the generalized companion matrix $C_0 C_1^{-1}$ (formed by* (5) *and* (6)*) of $P(\lambda)$ is multiplied on the right by the Hermitian symmetrizer* (22)*, the result is also Hermitian. A similar result holds in the complex symmetric case.*

Clearly, under the hypotheses of the theorem $\lambda H_0 - (C_0 C_1^{-1}) H_0$ is a Hermitian linearization of $P(\lambda)$. For cases when A_n is singular Hermitian linearizations can be found in [16].

3 Special degree-graded bases

As mentioned above, the family of degree-graded polynomials with recurrence relations of the form (2) include all the orthogonal bases, but is not limited to them. In this section, we discuss some well-known non-orthogonal bases of this kind and, consequently, for which the linearization $\lambda C_1 - C_0$ is strong.

3.1 Monomial basis

If in (2), we let $\alpha_j = 1$ and $\beta_j = \gamma_j = 0$, we get the monomial basis. Plugging these values into (14) and (15), we get:

$$E(\lambda) = \begin{bmatrix} I_s & & & & \\ \frac{1}{\lambda}I_s & I_s & & & \\ \frac{1}{\lambda^2}I_s & \frac{1}{\lambda}I_s & I_s & & \\ \vdots & \vdots & & \ddots & \\ \frac{1}{\lambda^{n-1}}I_s & \frac{1}{\lambda^{n-2}}I_s & \cdots & \frac{1}{\lambda}I_s & I_s \end{bmatrix}, \tag{23}$$

and

$$F_{i,j}(\lambda) = \begin{cases} \frac{1}{\lambda}I_s, & i = j = 1{:}(n-1) \\ \lambda^{n-1}I_s, & i = j = n \\ 0_s, & i = 1{:}(n-2); j = (i+1){:}(n-1) \\ -\sum_{k=0}^{n-2} A_k\lambda^k, & i = n-1; j = n \\ -\lambda^{n-i-1}\sum_{k=0}^{i-1} A_k\lambda^k, & i = (n-2){:}1; j = n, \end{cases} \tag{24}$$

and similarly from the fact that $H(\lambda) = E(\frac{1}{\lambda})$, we get:

$$H(\lambda) = \begin{bmatrix} I_s & & & & \\ \lambda I_s & I_s & & & \\ \lambda^2 I_s & \lambda I_s & I_s & & \\ \vdots & \vdots & & \ddots & \\ \lambda^{n-1}I_s & \lambda^{n-2}I_s & \cdots & \lambda I_s & I_s \end{bmatrix}, \tag{25}$$

and (21) gives $K(\lambda)$. In this case $Z_\phi = \{0\}$, and for λ near 0 block pivoting must be used [3].

3.2 Newton basis

Let an $s \times s$ matrix polynomial $P(\lambda)$ be specified by the data $\{(z_j, P_j)\}_{j=0}^{n}$ where the z_j's are distinct. Then, $P(\lambda)$ can be expressed in the Newton Basis. This basis has the following ordered form for $k = 0, \cdots, n$:

$$N_k(\lambda) = \prod_{j=0}^{k-1}(\lambda - z_j) \tag{26}$$

with $N_0(\lambda) = 1$. Therefore $Z_\phi = \{z_j\}_{j=0}^{n-1}$. Then the polynomial can be written in the form:

$$P(\lambda) = A_0N_0(\lambda) + A_1N_1(\lambda) + \cdots + A_nN_n(\lambda), \tag{27}$$

where the A_j's can be found either by divided differences or, equivalently, by solving this system:

$$\begin{bmatrix} I & & & & \\ I & N_1(z_1)I & & & \\ I & N_1(z_2)I & N_2(z_2)I & & \\ \vdots & \vdots & \vdots & \ddots & \\ I & N_1(z_n)I & N_2(z_n)I & \cdots & N_n(z_n)I \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}. \tag{28}$$

For more details see [3].

If in (2), we let $\alpha_j = 1$, $\beta_j = z_j$ and $\gamma_j = 0$, we get the Newton basis. Plugging these values into (14) and (15), we get:

$$E(\lambda) = \begin{bmatrix} I_s & & & & \\ \frac{1}{(\lambda-z_0)} I_s & I_s & & & \\ \frac{1}{(\lambda-z_0)(\lambda-z_1)} I_s & \frac{1}{(\lambda-z_1)} I_s & I_s & & \\ \vdots & \vdots & & \ddots & \\ \frac{1}{(\lambda-z_0)\cdots(\lambda-z_{n-2})} I_s & \frac{1}{(\lambda-z_1)\cdots(\lambda-z_{n-2})} I_s & \cdots & \frac{1}{(\lambda-z_{n-2})} I_s & I_s \end{bmatrix}, \tag{29}$$

and

$$F_{i,j}(\lambda) = \begin{cases} \frac{1}{z_{i-1}(\lambda-z_{i-1})} I_s, & i = j = 1{:}(n-1) \\ (z_0 \cdots z_{n-2})(\lambda - z_0)\cdots(\lambda - z_{n-2}) I_s, & i = j = n \\ -(z_0 \cdots z_{n-2}) \sum_{k=0}^{n-2} A_k N_k(\lambda), & i = n-1; j = n \\ -(z_0 \cdots z_{i-2} z_i \cdots z_{n-2})(\lambda - z_i)\cdots(\lambda - z_{n-2}) \sum_{k=0}^{i-1} A_k N_k(\lambda), & i = (n-2){:}1; j = n \\ 0_s, & \textit{otherwise}, \end{cases} \tag{30}$$

and similarly from the fact that $H(\lambda) = E(\frac{1}{\lambda})$, (21) gives $K(\lambda)$ and we also have

$$H(\lambda) = \begin{bmatrix} I_s & & & & \\ \frac{\lambda}{(1-\lambda z_0)} I_s & I_s & & & \\ \frac{\lambda^2}{(1-\lambda z_0)(1-\lambda z_1)} I_s & \frac{\lambda}{(1-\lambda z_1)} I_s & I_s & & \\ \vdots & \vdots & & \ddots & \\ \frac{\lambda^{n-1}}{(1-\lambda z_0)\cdots(1-\lambda z_{n-2})} I_s & \frac{\lambda^{n-2}}{(1-\lambda z_1)\cdots(1-\lambda z_{n-2})} I_s & \cdots & \frac{\lambda}{(1-\lambda z_{n-2})} I_s & I_s \end{bmatrix}. \tag{31}$$

3.3 Pochhammer basis

The Pochhammer basis is just a special Newton basis with nodes $z_j = -(a + j)$, $j = 0, \ldots, n-1$. The Pochhammer basis is used in combinatorial applications and in the solution of difference equations. Some good sparse polynomial interpolation algorithms have been developed using this basis (see [20], for example). If in (2), we let $\alpha_j = 1$, $\beta_j = -(a + j)$ and $\gamma_j = 0$, then the Pochhammer basis is generated.

4 Interpolating with Bernstein polynomials

Bernstein Polynomials have the form:

$$b_{j,n}(\lambda; a, b) = \frac{1}{(b-a)^n} \binom{n}{j} (\lambda - a)^j (b - \lambda)^{n-j} \tag{32}$$

for $n = 1, 2, \cdots$ and $j = 0, 1, \cdots, n$, and have good (uniform) convergence properties to continuous functions on (a, b) (see [8]). They are widely used in geometric computing (see [9] and [10]) and, clearly, they are not degree-graded. Here $Z_\phi = \{a, b\}$ contains only two elements.

4.1 Linearization

An $s \times s$ matrix polynomial $P(\lambda)$ of degree n can be written in terms of a set of Bernstein polynomials:

$$P(\lambda) = A_n b_{n,n}(\lambda; a, b) + A_{n-1} b_{n-1,n}(\lambda; a, b) + \ldots + A_1 b_{1,n}(\lambda; a, b) + A_0 b_{0,n}(\lambda; a, b). \quad (33)$$

For convenience, let us assume $n = 5$ and the generalizations for all positive n will be clear. Define block-matrices

$$C_0 = \begin{bmatrix} \frac{5a}{b-a} I_s & 0 & 0 & 0 & -\frac{b}{b-a} A_0 \\ \frac{b}{b-a} I_s & \frac{4a}{2(b-a)} I_s & 0 & 0 & -\frac{b}{b-a} A_1 \\ 0 & \frac{b}{b-a} I_s & \frac{3a}{3(b-a)} I_s & 0 & -\frac{b}{b-a} A_2 \\ 0 & 0 & \frac{b}{b-a} I_s & \frac{2a}{4(b-a)} I_s & -\frac{b}{b-a} A_3 \\ 0 & 0 & 0 & \frac{b}{b-a} I_s & \frac{a}{5(b-a)} A_5 - \frac{b}{b-a} A_4 \end{bmatrix}, \quad (34)$$

$$C_1 = \begin{bmatrix} \frac{5}{b-a} I_s & 0 & 0 & 0 & -\frac{1}{b-a} 4A_0 \\ \frac{1}{b-a} I_s & \frac{4}{2(b-a)} I_s & 0 & 0 & -\frac{1}{b-a} A_1 \\ 0 & \frac{1}{b-a} I_s & \frac{3}{3(b-a)} I_s & 0 & -\frac{1}{b-a} A_2 \\ 0 & 0 & \frac{1}{b-a} I_s & \frac{2}{4(b-a)} I_s & -\frac{1}{b-a} A_3 \\ 0 & 0 & 0 & \frac{1}{b-a} I_s & \frac{1}{5(b-a)} A_5 - \frac{1}{b-a} A_4 \end{bmatrix}. \quad (35)$$

For more details see [17, 23, 1].

A little computation shows that

$$\left(\begin{bmatrix} b_{0,5}(\lambda; a, b) & b_{1,5}(\lambda; a, b) & b_{2,5}(\lambda; a, b) & b_{3,5}(\lambda; a, b) & b_{4,5}(\lambda; a, b) \end{bmatrix} \otimes I_s\right)(\lambda C_1 - C_0)$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{b-\lambda}{b-a} P(\lambda) \end{bmatrix}.$$

This is an obvious analogue of equation (7) for degree-graded polynomials. As in that case, it can be seen that $\lambda C_1 - C_0$ and $P(\lambda)$ have the same eigenvalues. For $\lambda \in Z_\phi$ again block pivoting can be used; but in this case it turns out that we may cover the case $\lambda = b$ together with all $\lambda \neq a$ (in practice we would use the block pivoting if λ is near to a, not just equal to it). An analogue of Theorem 2 also holds:

Theorem 4 *Let $P(\lambda)$ be a matrix polynomial of degree n and $\{b_{i,n}(\lambda; a, b)\}_{i=0}^n$ be a system of Bernstein polynomials. If $\lambda = a$ is not an eigenvalue of $P(\lambda)$, then the pencil $\lambda C_1 - C_0$ defined by* (34) *and* (35) *is a strong linearization of $P(\lambda)$. If $\lambda = a$ is an eigenvalue, then block pivoting can be used to get a strong linearization.*

Proof: The proof is very similar to the proof of Theorem 2, so we give only a brief outline. In [3], the λ-dependent LU factors of $\lambda C_1 - C_0$ corresponding to a pencil of the form (34)– (35) and of degree n are explicitly given as follows:

$$L(\lambda) = \begin{bmatrix} I_s & & & \\ -\frac{b-\lambda}{n(\lambda-a)} I_s & I_s & & \\ & \ddots & \ddots & \\ & & -\frac{(n-1)(b-\lambda)}{2(\lambda-a)} I_s & I_s \end{bmatrix}, \quad (36)$$

$$U(\lambda) = \begin{bmatrix} \frac{n(\lambda-a)}{b-a} I_s & & & & U_{1,n}(\lambda) \\ & \ddots & & & \vdots \\ & & \frac{3(\lambda-a)}{(n-2)(b-a)} I_s & & U_{n-2,n}(\lambda) \\ & & & \frac{2(\lambda-a)}{(n-1)(b-a)} I_s & U_{n-1,n}(\lambda) \\ & & & & U_{n,n}(\lambda) \end{bmatrix}, \tag{37}$$

where

$$U_{i,n}(\lambda) = \begin{cases} \frac{b-\lambda}{b-a} A_0, & i = 1 \\ \frac{b-\lambda}{b-a} A_{j-1} + \frac{(j-1)(b-\lambda)}{(n-j+2)(\lambda-a)} U_{j-1,n}(\lambda), & i = 2{:}(n-1) \\ \frac{(b-a)^{n-1}}{n(\lambda-a)^{n-1}} P(\lambda). & i = n \end{cases} \tag{38}$$

As in the degree-graded case (Theorem 2), we now replace the last block entry of (37) by

$$\widetilde{U}_{n,n}(\lambda) = \frac{b_{0,n}(\lambda; a, b)(b-a)^{n-1}}{b_{n-1,n}(\lambda; a, b)(b-\lambda)^{n-1}} I_s = \frac{(b-a)^{n-1}}{n(\lambda-a)^{n-1}} I_s, \tag{39}$$

Moreover, looking at (15), we only need α_j, γ_j and k_{n-1} and k_n to construct $F(\lambda)$. By comparison, it turns out that in this case $\alpha_j(\lambda) = \frac{b-\lambda}{b-a}$, $\gamma_j(\lambda) = 0$ and $k_{n-1}(\lambda) = \frac{b-\lambda}{b-a}$. Here, as opposed to (2), α and k_{n-1} are λ-dependent, and (3) is no longer valid. Now, we can compute a unimodular matrix $F(\lambda)$ analogous to (15).

For the reverse case, instead of (20), we now use

$$\widetilde{U_1}_{n,n}(\lambda) = \frac{(b-a)^{n-1}}{n(1-\lambda a)^{n-1}} I_s. \tag{40}$$

The rest of the proof is exactly the same as that of Theorem 2. □

4.2 Symmetrizing the linearization

The idea discussed in Section 2.2 applies to the Bernstein case as well. Indeed, the following proposition is easily verified:

Proposition 5 *Let* $\{b_{i,n}(\lambda; a, b)\}_{i=0}^n$ *be a system of Bernstein polynomials as in* (32)*. Moreover, let* $P(\lambda)$ *be a Hermitian matrix polynomial defined in that basis. Then, when the generalized companion matrix* $C_0 C_1^{-1}$ *(formed by* (34) *and* (35)*) of* $P(\lambda)$ *is multiplied on the right by the Hermitian symmetrizer* (22)*, the result is also Hermitian.*

5 Interpolating with Lagrange polynomials

5.1 Linearization

Lagrange polynomial interpolation is traditionally viewed as a tool for theoretical analysis; however, recent work reveals several advantages to computation in the Lagrange basis (see e.g. [7, 15]). As above, suppose that an $s \times s$ matrix polynomial $P(\lambda)$ of degree n is sampled at $n+1$ distinct points $z_0, z_1, \ldots, z_n$, and write $P_j := P(z_j)$. Lagrange polynomials are defined by

$$\ell_j(\lambda) = w_j \prod_{k=0,\, k\neq j}^{n} (\lambda - z_j), \qquad j = 0, 1, \ldots, n \tag{41}$$

(and so $Z_\phi = \{z_j\}_{j=0}^n$) where the "weights" w_j are

$$w_j = \prod_{k=0,\, k\neq j}^{n} \frac{1}{z_j - z_k}. \tag{42}$$

Then $P(\lambda)$ can be expressed in terms of its samples in the form $P(\lambda) = \sum_{j=0}^n \ell_j(\lambda) P_j$.

The companion pencil $\lambda C_1 - C_0$ as formulated in Section 3.2 of [1], or equations (4.5) of [2], has (when $n = 3$):

$$\lambda C_1 - C_0 = \begin{bmatrix} (\lambda - z_0)I & 0 & 0 & 0 & -P_0 \\ 0 & (\lambda - z_1)I & 0 & 0 & -P_1 \\ 0 & 0 & (\lambda - z_2)I & 0 & -P_2 \\ 0 & 0 & 0 & (\lambda - z_3)I & -P_3 \\ w_0 I & w_1 I & w_2 I & w_3 I & 0 \end{bmatrix}. \tag{43}$$

The extension to general n is obvious.

Let us define a polynomial $\hat{P}(\lambda)$ by the (apparently) trivial device of adding terms in λ^{n+1} and λ^{n+2} with zero matrix coefficients to $P(\lambda)$ (see [12]). This introduces infinite eigenvalues that are defective. The following result then determines the nature of the infinite eigenvalue of $P(\lambda)$ via that of the zero eigenvalue of $\hat{P}^\sharp(\lambda)$.

Proposition 6 *Let $P(\lambda) = \sum_{j=0}^n A_j \lambda_j$ with $\det(A_n) = 0$, $A_n \neq 0$, so that $P(\lambda)$ has an infinite eigenvalue. If this infinite eigenvalue of $P(\lambda)$ has partial multiplicities $m_1 \geq \cdots \geq m_t > 0$ then $t = n - \mathtt{rank}(A_n)$ and $\hat{P}(\lambda)$ has an infinite eigenvalue with partial multiplicities $m_1 + 2, \cdots, m_t + 2,\ 2, \cdots, 2$ (the "2" being repeated $n - t$ times).*

Proof: The partial multiplicities of the eigenvalues of $P(\lambda)$ at infinity coincide with those of the zero eigenvalue of $P^\sharp(\lambda) = \lambda^n P(\frac{1}{\lambda})$. By Theorem A.3.4 of [14]

$$P^\sharp(\lambda) = E_0(\lambda)\ \mathtt{diag}\left[\ \lambda^{m_1}, \quad \cdots, \quad \lambda^{m_t}, \quad 1, \quad \cdots, \quad 1\ \right] F_0(\lambda) \tag{44}$$

for matrix polynomials $E_0(\lambda), F_0(\lambda)$ invertible at 0 and since $P^\sharp(0) = A_n$, it follows that $n - t = \mathtt{rank}(A_n)$, or $t = n - \mathtt{rank}(A_n)$.

For the reverse polynomial of $\hat{P}(\lambda)$,

$$\hat{P}^\sharp(\lambda) = \lambda^n \hat{P}(\frac{1}{\lambda}) = \lambda^{n+2} P(\frac{1}{\lambda}) = \lambda^2(\lambda^n P(\frac{1}{\lambda})) = \lambda^2 P^\sharp(\lambda). \tag{45}$$

It follows from (44) that

$$\hat{P}^\sharp(\lambda) = E_0(\lambda)\ \mathtt{diag}\left[\ \lambda^{m_1+2}, \quad \cdots, \quad \lambda^{m_t+2}, \quad \lambda^2, \quad \cdots, \quad \lambda^2\ \right] F_0(\lambda). \tag{46}$$

But this is just a Smith form for $\hat{P}^\sharp(\lambda)$ and shows that $\hat{P}(\lambda)$ itself has an infinite eigenvalue with the multiplicities claimed. □

Theorem 7 *The pencil $\lambda C_1 - C_0$ of equation (43) is a strong linearization of $\hat{P}(\lambda)$.*

Proof: Again the proof is very similar to the proof of Theorem 2. Assume first that Z_ϕ does not intersect the set of all eigenvalues of $P(\lambda)$ and $P^\sharp(\lambda)$.

In [3], the λ-dependent LU factors of $\lambda C_1 - C_0$ corresponding to a pencil of the form (43) and of degree n are explicitly given as follows:

$$L = \begin{bmatrix} I_s & & & \\ & \ddots & & \\ & & I_s & \\ \frac{w_0}{\lambda - z_0} I_s & \cdots & \frac{w_n}{\lambda - z_n} I_s & I_s \end{bmatrix}, \tag{47}$$

$$U = \begin{bmatrix} (\lambda - z_0) I_s & & & -P_0 \\ & \ddots & & \vdots \\ & & (\lambda - z_n) I_s & -P_n \\ & & & \frac{1}{(\lambda - z_0)\cdots(\lambda - z_n)} P(\lambda) \end{bmatrix}. \tag{48}$$

Here, to get $\widetilde{U}(\lambda)$, we replace the last block entry of (48) by

$$\widetilde{U}_{n+2,n+2}(\lambda) = \frac{1}{(\lambda - z_0)\cdots(\lambda - z_n)} I_s. \tag{49}$$

It turns out that

$$E(\lambda) = \begin{bmatrix} I_s & & & & \\ & I_s & & & \\ & & I_s & & \\ & & & \ddots & \\ -\frac{w_0}{\lambda - z_0} I_s & -\frac{w_1}{\lambda - z_1} I_s & \cdots & -\frac{w_n}{\lambda - z_n} I_s & I_s \end{bmatrix}, \tag{50}$$

$$F_{i,j}(\lambda) = \begin{cases} \frac{1}{\lambda - z_{i-1}} I_s, & i = j = 1{:}(n+1) \\ (\lambda - z_0)\cdots(\lambda - z_n) I_s, & i = j = n+2 \\ (\lambda - z_0)\cdots(\lambda - z_{i-2})(\lambda - z_i)\cdots(\lambda - z_n)\hat{P}_{i-1}, & i = 1{:}(n+1); j = n+2 \\ 0_s, & \textit{otherwise}, \end{cases} \tag{51}$$

where $\hat{P}_i$ are the values of $\hat{P}(\lambda)$ evaluated at the nodes.

For the reverse case, we use

$$\widetilde{U1}_{n+2,n+2}(\lambda) = \frac{1}{(1 - \lambda z_0)\cdots(1 - \lambda z_n)} I_s, \tag{52}$$

instead of (20). As with the other bases, we can use (51) to construct $K(\lambda)$:

$$K_{i,j}(\lambda) = \begin{cases} \frac{F_{i.j}(\frac{1}{\lambda})}{\lambda}, & i = j = 1{:}(n+1) \\ \lambda^{n+1} F_{i,n+2}(\frac{1}{\lambda}), & i = 1{:}(n+2); j = n+2 \end{cases} \tag{53}$$

The rest of the proof is exactly the same as the proof of Theorem 2 (including the reference to [3] for the block pivoting case). □

Remark. Computation of the right eigenvectors of the pencil (43) allows one to recover the right eigenvectors of $P(\lambda)$ in the following manner: the right eigenvectors of the pencil (43) are of the form $[\ell_0(\lambda)v, \ell_1(\lambda)v, \ldots, \ell_n(\lambda)v, 0]^T$, and since $1 = \sum_{k=0}^{n} \ell_k(\lambda)$, simply adding these subvectors gives v (see [1] for details). The numerical stability of this procedure has not been established.

5.2 Symmetrizing the Lagrangian companion pencil

Multiplying $\lambda C_1 - C_0$ of (43) on the right by the block-diagonal

$$A := \begin{bmatrix} w_0^{-1}P_0 & 0 & 0 & 0 \\ 0 & w_1^{-1}P_1 & 0 & 0 \\ 0 & 0 & w_2^{-1}P_2 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}$$

we obtain

$$(\lambda C_1 - C_0)A = \begin{bmatrix} \frac{\lambda - z_0}{w_0}P_0 & 0 & 0 & P_0 \\ 0 & \frac{\lambda - z_1}{w_1}P_1 & 0 & P_1 \\ 0 & 0 & \frac{\lambda - z_2}{w_2}P_2 & P_2 \\ P_0 & P_1 & P_2 & 0 \end{bmatrix}.$$

As in Section 2.2, the reason for doing this is that the pencil on the right is now **block-symmetric**. This can provide computational advantages, but it is particularly interesting when, as in many applications, the z_j (and hence w_j) are real and $P_0, \ldots, P_n$ are Hermitian ($P_j^H = P_j$), or when the data are complex symmetric ($P_j^T = P_j$).

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