ON THE ZEROS OF COSINE POLYNOMIALS: SOLUTION TO A PROBLEM OF LITTLEWOOD

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ABSTRACT. Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [12, problem 22] poses the following research problem, which appears to still be open:

Problem. *"If the n_j are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less."*

No progress appears to have been made on this in the last half century. We show that this is false.

Theorem. *There exists a cosine polynomial $\sum_{j=1}^{N} \cos(n_j\theta)$ with the n_j integral and all different so that the number of its real zeros in the period is $O\left(N^{9/10}(\log N)^{1/5}\right)$.*

1. Littlewood's 22nd Problem

Problem. *"If the n_j are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less."*

Here "real zeros" means "zeros in a period". Denote the number of zeros of a trigonometric polynomial T in the period $[-\pi, \pi)$ by $\mathcal{N}(T)$.

Note that if T is a real trigonometric cosine polynomial of degree n, then it is of the form $T(t) = \exp(-int)P(\exp(it))$, $t \in \mathbb{R}$, where P is a reciprocal algebraic polynomial of degree $2n$, and if T has only real zeros, then P has all its zeros on the unit circle. So in terms or reciprocal algebraic polynomials one is looking for a reciprocal algebraic polynomial with coefficients in $\{0,1\}$, with $2N$ terms, and with $N-1$ or fewer zeros on the unit circle. Even achieving $N-1$ is fairly hard. An exhaustive search up to degree $2N = 32$ yields only 10 example achieving $N-1$ and only one example with fewer.

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This first example disproving the "possibly $N-1$" part of the conjecture is

$$\sum_{j=0,\, j\notin\{9,10,11,14\}}^{14} (z^j + z^{28-j})$$

which has 8 roots of modulus 1 and corresponds to a cosine sum of 11 terms with 8 roots in the period.

It is hard to see how one might generate infinitely many such examples or indeed why Littlewood made his conjecture.

The following is a reciprocal polynomial with 32 terms and exactly 14 zeros of modulus 1:

$$\sum_{j=0,\, j\notin\{10,11,17,19\}}^{19} (z^j + z^{38-j})\,.$$

So it corresponds to a cosine sum of 16 terms with 14 zeros in $[-\pi, \pi)$. In other words the sharp version of Littlewood's conjecture is false again, though barely. The following is a reciprocal polynomial with 280 terms and 52 zeros of modulus 1:

$$\sum_{j=0,\, j\notin\{124,125,126,127,128,134,141,143,145,147,148,151,152\}}^{152} (z^j + z^{304-j})\,.$$

So it corresponds to a cosine sum of 140 terms with 52 zeros in $[-\pi, \pi)$. Once again the sharp version of Littlewood's conjecture is false, though this time by a margin. It was found by a version of the greedy algorithm (and some guessing). There is no reason to believe it is a minimal example.

The interesting feature of this example is how close it is to the Dirichlet kernel $(1 + z + z^2 + \ldots + z^{304})$. This is not accidental and suggests the approach that leads to our main result.

Littlewood explored many problems concerning polynomials with various restrictions on the coefficients. See [9], [10], and [11], and in particular Littlewood's delightful monograph [12]. Related problems and results may be found in [2] and [4], for example. One of these is Littlewood's well-known conjecture of around 1948 asking for the minimum L_1 norm of polynomials of the form

$$p(z) := \sum_{j=0}^{n} a_j z^{k_j}\,,$$

where the coefficients a_j are complex numbers of modulus at least 1 and the exponents k_j are distinct nonnegative integers. It states that such polynomials have L_1 norms on the unit circle that grow at least like $c \log n$. This was proved by S. Konyagin [7] and independently by McGehee, Pigno, and Smith [13] in 1981. A short proof is available in [5]. It is believed that the minimum, for polynomials of degree n with complex coefficients of modulus at least 1 is attained by $1 + z + z^2 + \cdots + z^n$, but this is open.

2. Auxiliary Functions

The key is to construct n term cosine sums that are large most of the time. This is the content of this section.

Lemma 1. *There is an absolute constant c_1 such that for all n and $\alpha > 1$ there are coefficients $a_0, a_1, \dots, a_n$ with each $a_j \in \{0,1\}$ such that*

$$\text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \le \alpha\} \le c_1 \alpha n^{-1/2},$$

where

$$P_n(t) = \sum_{j=0}^{n} a_j \cos(jt).$$

Proof. We will prove the stronger result that there is an absolute constant c_1 such that for all $\alpha > 0$ and all n

$$\lambda(\alpha) := 2^{-(n+1)} \sum_{\{a_0, a_1, \dots, a_n\}} \text{meas}\{t \in [-\pi, \pi) : |P_n(t)| \le \alpha\} \le c_1 \alpha n^{-1/2}.$$

If $X_0, X_1, \dots, X_n$ are independent Bernoulli random variables with

$$P(X_j = 0) = P(X_j = 1) = \frac{1}{2}, \qquad j = 0, 1, \dots, n,$$

then the indicated average is an expected value. Let

$$R_n(t) = \sum_{j=0}^{n} X_j \cos(jt)$$

and note that

$$\lambda(\alpha) = \int_{-\pi}^{\pi} P(|R_n(t)| \le \alpha)\, dt.$$

Define

$$D_n(t) := \sum_{j=0}^{n} \cos(jt).$$

The expected value of $R_n(t)$ is $\mu_n(t) := D_n(t)/2$; its variance is

$$\sigma_n^2(t) := \frac{1}{4} \sum_{0}^{n} \cos^2(jt) = \frac{1}{8}(n + 1 + D_n(2t)).$$

We now apply a uniform normal approximation to get the desired result. Define the cumulative normal distribution function by

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du.$$

Define

$$\varrho_2 := \frac{1}{n+1}\sum_{j=0}^{n} \mathrm{Var}(X_j\cos(jt)) =$$

$$= \frac{1}{4(n+1)}\sum_{0}^{n}\cos^2(jt) = \frac{1}{8}\left(1+\frac{D_n(2t)}{n+1}\right),$$

$$\varrho_3 := \frac{1}{n+1}\sum_{j=0}^{n} \mathrm{E}\left(\left|\left(X_j-\frac{1}{2}\right)\cos(jt)\right|^3\right)$$

We suppress the dependence of each of these on n and u. The Berry-Esseen bound in Bhattacharya and Ranga Rao [1, Theorem 12.4, page 104] is that

$$\left|P(R_n(t)\le c) - \Phi\left(\frac{c-\mu_n(t)}{\sigma_n(t)}\right)\right| \le \frac{11\varrho_3}{4\sqrt{n}\,\varrho_2^{3/2}}.$$

It is elementary that $\varrho_3 \le 1/8$. Moreover there is an absolute constant $c_2 > 0$ such that $\varrho_2 > c_2$ for all $t \in \mathbb{R}$ and all $n = 1, 2, \ldots$. Finally the function Φ has derivative bounded by $(2\pi)^{-1/2}$ so

$$|\Phi(x)-\Phi(y)| \le (2\pi)^{-1/2}|x-y|, \qquad x, y \in \mathbb{R}.$$

It follows that there is an absolute constant c_1 such that

$$P(-\alpha \le R_n(u) \le \alpha) \le c_1\alpha n^{-1/2}.$$

□

3. The Main Theorem

Theorem 1. *There exists a cosine polynomial* $\sum_{j=1}^{N}\cos(n_j\theta)$ *with the* n_j *integral and all different so that the number of its real zeros in the period is*

$$O\left(N^{9/10}(\log N)^{1/5}\right).$$

We note that we have not worked hard to replace the exponent 9/10 with a smaller one that we may call a "close to optimal" exponent in the result. One can hope to replace the exponent 9/10 in Theorem 1 by a slightly smaller one.

The proof of our main theorem above follows immediately from the following Lemma 2 stated below and Lemma 1. Namely, take $m := N+1$, $n = m^{2/5}(\log m)^{-4/5}$, $\alpha = n^{1/4}$ and $\beta = c_1\alpha n^{-1/2} = c_1 n^{-1/4}$.

Lemma 2. *Let* $n \le m$,

$$D_m(t) := \sum_{j=0}^{m}\cos(jt),$$

$$P_n(t) := \sum_{j=0}^{n} a_j\cos(jt), \qquad a_j \in \{0,1\}.$$

Suppose $\alpha \ge 1$ *and*

$$\operatorname{meas}\{t \in [-\pi,\pi) : |P_n(t)| \le \alpha\} \le \beta\,.$$

Let $S_m := D_m - P_n$. *Then the number of zeros of* S_m *in* $[-\pi,\pi)$ *is at most*

$$\frac{c_3 m}{\alpha} + c_4 m\beta + c_5 n m^{1/2} \log m\,,$$

where c_3, c_4, *and* c_5 *are absolute constants.*

To prove Lemma 2 we need the following consequence of the Erdős-Turán Theorem [15, p. 278]; see also [6].

Lemma 3. *Let*

$$S_m(t) = \sum_{j=0}^{m} a_j \cos(jt)\,, \qquad a_j \in \{0,1\}\,,$$

be not identically zero. Denote the number of zeros of S_m *in an interval* $I \subset [-\pi,\pi)$ *by* $\mathcal{N}(I)$. *Then*

$$\mathcal{N}(I) \le c_6 m|I| + c_6\sqrt{m}\log m\,,$$

where c_6 *is an absolute constant and* $|I|$ *denotes the length of* I.

Now we prove Lemma 2.

Proof. We write

$$\{t \in [-\pi,\pi) : |P_n(t)| \le \alpha\} = \bigcup_{j=1}^{k} I_j\,,$$

where the intervals I_j are disjoint and $k \le 2n$. Let

$$I_0 := \{t \in [-\pi,\pi) : |D_m(t)| \ge \alpha\}\,.$$

Note that $I_0 \subset [-c/\alpha, c/\alpha]$. Then S_m has all its zeros in $\bigcup_{j=0}^{k} I_j$. By Lemma 3 we have

$$\mathcal{N}(I_j) \le c_6 m|I_j| + c_6\sqrt{m}\log m\,, \qquad j = 1,2,\dots,k\,,$$

and

$$\mathcal{N}(I_0) \le c_6 m|I_0| + c_6\sqrt{m}\log m \le \frac{c_7 m}{\alpha} + c_7\sqrt{m}\log m$$

with an absolute constant c_7. So

$$\begin{aligned}
\mathcal{N}([-\pi,\pi)) \le& \sum_{j=0}^{k} \mathcal{N}(I_j) \\
\le& \frac{c_7 m}{\alpha} + c_7\sqrt{m}\log m + c_6\sum_{j=1}^{k} m|I_j| + kc_7\sqrt{m}\log m \\
\le& \frac{c_7 m}{\alpha} + c_6 m\beta + 2nc_7\sqrt{m}\log m
\end{aligned}$$

and the proof is finished. □

4. Average Number of Real Zeros

Why did Littlewood make this conjecture? He might have observed that the average number of zeros of a trigonometric polynomial of the form

$$0 \neq T(t) = \sum_{j=1}^{n} a_j \cos(jt)\,, \qquad a_j \in \{0,1\}\,,$$

has in $[0, 2\pi)$ is at least cn. This is what we elaborate in this section. Associated with a polynomial P of degree exactly n with real coefficients we introduce $P^*(z) := z^n P(1/z)$.

Theorem 2. *Let*

$$S(t) := \sum_{j=1}^{n} a_j \cos(jt) \qquad \text{and} \qquad \widetilde{S}(t) := \sum_{j=1}^{n} a_{n+1-j} \cos(jt)\,,$$

where each of the coefficients a_j is real and $a_1 a_n \neq 0$. Let w_1 be the number of zeros of S in $[0, 2\pi)$, and let w_2 be the number of zeros of $\widetilde{S}$ in $[0, 2\pi)$. Then $w_1 + w_2 \geq 2n$.

Proof. Let $P(z) = \sum_{j=1}^{n} a_j z^j$. Without loss of generality we may assume that P does not have zeros on the unit circle; the general case follows by a simple limiting argument with the help of Rouché's Theorem. Note that if P has exactly k zeros in the open unit disk then $zP^*(z)$ has exactly $n - k$ zeros in the open unit disk. Also,

$$2S(t) = \operatorname{Re}(P(e^{it})) \qquad \text{and} \qquad 2\widetilde{S}(t) = \operatorname{Re}(e^{it} P^*(e^{it}))\,.$$

Hence the theorem follows from the Argument Principle. Note that if a continuous curve goes around the origin k times then it crosses the real axis at least $2k$ times. □

Theorem 2 has some interesting consequences. As an example we can state and easily see the following.

Theorem 3. *The average number of zeros of trigonometric polynomials in the class*

$$\left\{ \sum_{j=1}^{n} a_j \cos(jt), \;\; a_j \in \{-1, 1\} \right\}$$

in $[0, 2\pi)$ is at least n. The average number of zeros of trigonometric polynomials in the class

$$\left\{ 0 \neq \sum_{j=1}^{n} a_j \cos(jt), \;\; a_j \in \{0, 1\} \right\}$$

in $[0, 2\pi)$ is at least $n/4$.

Proof. Most of the cosine sums in both classes naturally break into pairs with a large combined total number of real zeros in the period. □

5. Conclusion

Let $0 \le n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T_n(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the Littlewood's Conjecture proved by S. Konyagin [7] and independently by McGehee, Pigno, and Smith [13] in 1981. See also [5] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T_n((2j-1)\pi/n_N) = 0\,.$$

See [8], for example. Another way is to use Theorem 2 of [14]. So there is certainly no shortage of possible approaches to prove the starting observation of our conclusion even in the case $n_1 = 0$. It seems likely that the number of zeros of the above sums in a period must tend to infinity with N. This does not appear to be easy. The case when the sequence $0 \le n_0 \le n_1 \le \cdots$ is fixed will be handled in a forthcoming paper [3].

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