The Average Norms of Polynomials

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Average Norms of Polynomials

Let $n \geq 0$ be any integer and

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = 0, \pm 1 \right\}$$

be the polynomials of height 1 and degree n. Let

$$\beta_n(m) := \frac{1}{3^{n+1}} \sum_{P \in \mathcal{F}_n} \|P\|_m^m.$$

Here $\|P\|_m^m$ is the mth power of the L_m norm on the boundary of the unit disc.

So $\beta_n(m)$ is the average of the mth power of the L_m norm over $\mathcal{F}_n$.

Typical of the results we get is is:

Theorem *For $n \geq 0$, we have*

$$\beta_n(2) = \frac{2}{3}(n+1),$$

$$\beta_n(4) = \frac{8}{9}n^2 + \frac{14}{9}n + \frac{2}{3}$$

and

$$\beta_n(6) = \frac{16}{9}n^3 + 4n^2 + \frac{26}{9}n + \frac{2}{3}.$$

The Littlewood polynomials $\mathcal{L}_n$ are defined as follows:

$$\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = \pm 1 \right\}.$$

Now let

$$\mu_n(m) := \frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_n} \|P\|_m^m$$

be the average of the mth power of the L_m norms over $\mathcal{L}_n$.

Here

$$\|P\|_m := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^m d\theta \right\}^{\frac{1}{m}}, \quad (z = e^{i\theta})$$

We are interested in finding exact formulae for $\mu_n(m)$.

Theorem *For* $n \geq 0$*, we have*

$$\mu_n(2) = n + 1,$$

$$\mu_n(4) = 2n^2 + 3n + 1,$$

$$\mu_n(6) = 6n^3 + 9n^2 + 4n + 1$$

and

$$\mu_n(8) =$$

$$24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n.$$

That $\mu_n(2) = n+1$ is trivial since $\|P\|_2 = n+1$ for each $P \in \mathcal{L}_n$.

The above result for $\mu_n(4)$ is due to Newman and Byrnes.

The results for $\mu_n(6)$ and $\mu_n(8)$ are new and are the tip of an iceberg.

What is striking, and perhaps surprising, is that such exact formulae exist at all.

One interesting generalization is the following. Let

$$\mathcal{F}_n(H) := \left\{ \sum_{i=0}^{n} a_i z^i : |a_i| \le H, a_i \in \mathbb{Z} \right\}$$

be the set of all the polynomials of height H of degree $\le n$. Let

$$\beta_n(m, H) := \frac{1}{(2H+1)^{n+1}} \sum_{P \in \mathcal{F}_n(H)} \|P\|_m^m$$

Then

$$\begin{aligned}\beta_n(4, H) = \frac{2}{9} H^2 (H+1)^2 n^2 \\ + \frac{1}{45} H(H+1)(19H^2 + 19H - 3)n \\ + \frac{1}{15} H(H+1)(3H^2 + 3H - 1).\end{aligned}$$

There are many limiting results concerning expected norms.

The expected norms of random Littlewood polynomials, q_n of degree n, satisfy

$$\frac{\mathsf{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1+p/2))^{1/p}$$

and for their derivatives

$$\frac{\mathsf{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

There is a considerable literature on the maximum and minimum norms of polynomials in $\mathfrak{L}_n$. In the L_4 norm this problem is often called Golay's "Merit Factor" problem.

The specific old and difficult problem is to find the minimum possible L_4 norm of a polynomial in $\mathcal{L}_n$. The cognate problem in the supremum norm is due to Littlewood.

Both of these problems are at least 50 years old and neither is solved.

As before let $n \geq 0$ be any integer and

$$\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = \pm 1 \right\}$$

be the set of all Littlewood polynomials of degree n. Let

$$\mu_n(m) := \frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_n} \|P\|_m^m$$

be the average of the mth power of the L_m norm over $\mathcal{L}_n$. We are interested in finding exact formulae for $\mu_n(m)$.

For any complex number z on the unit circle and any real number h, we have

$$|z+h|^2+|z-h|^2=2(|z|^2+h^2)$$

and

$$|z+h|^4+|z-h|^4=$$

$$2(|z|^4+4h^2|z|^2+h^4+h^2(z^2+\overline{z}^2))$$

With similar more complicated expressions for sixth powers and eighth powers.

Hence for any polynomial $P(z)$,

$$\|zP(z)+h\|_2^2+\|zP(z)-h\|_2^2 = 2(\|P(z)\|_2^2+h^2)$$

and

$$\|zP(z)+h\|_4^4+\|zP(z)-h\|_4^4 =$$

$$2(\|P(z)\|_4^4+4h^2\|P(z)\|_2^2+h^4)$$

We also have, for induction purposes,

$$\sum_{P\in\mathcal{L}_n}\|P\|_m^m =$$

$$\sum_{P\in\mathcal{L}_{n-1}}\left(\|zP(z)+1\|_m^m+\|zP(z)-1\|_m^m\right).$$

Theorem *For $n \geq 0$, we have*

$$\mu_n(2) = n + 1,$$

$$\mu_n(4) = 2n^2 + 3n + 1$$

and

$$\mu_n(6) = 6n^3 + 9n^2 + 4n + 1.$$

The first formula is trivial because every Littlewood polynomial of degree n has constant L_2 norm $\sqrt{n+1}$. However we give a simple inductive proof because it is indicative of the basic method behind all the proofs.

Using the above formulae with $m = 2$ for the 2 norm, we have

$$\mu_n(2) =$$

$$\frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_{n-1}} \left(\|zP(z) + 1\|_2^2 + \|zP(z) - 1\|_2^2 \right)$$

$$= \frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_{n-1}} 2(\|P\|_2^2 + 1)$$

$$= \mu_{n-1}(2) + 1$$

for any $n \geq 1$. It is clear that $\mu_0(m) = 1$ for any m. Thus we have

$$\mu_n(2) = \mu_0(2) + n = n + 1.$$

With $m = 4$, for the 4 norm

$$\begin{aligned}\mu_n(4) &= \frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_{n-1}} 2\left(\|P\|_4^4 + 4\|P\|_2^2 + 1\right) \\ &= \mu_{n-1}(4) + 4\mu_{n-1}(2) + 1\end{aligned}$$

for any $n \geq 1$.

For the 6 norm we need two lemmas.

Lemma *For $m \neq 0$, we have*

$$\sum_{P \in \mathcal{L}_n} \int_0^{2\pi} |P(z)|^2 z^m d\theta = 0.$$

Lemma *For $m \geq 1$, we have*

$$\sum_{P \in \mathcal{L}_n} \int_0^{2\pi} |P(z)|^2 z^m P(z)^2 d\theta = 0.$$

Let $n \geq 0$ be any integer and

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = 0, \pm 1 \right\}$$

be the set of all the polynomials of height 1 of degree n. Let

$$\beta_n(m) := \frac{1}{3^{n+1}} \sum_{P \in \mathcal{F}_n} \|P\|_m^m.$$

We can also obtain exact formulae for $\beta_n(m)$. The additional details involve observing that since

$$\|zP(z) + 0\|_m^m = \|P(z)\|_m^m$$

The previous equations can be extended easily to allow summing over all the height one polynomials. For example, for any polynomial $P(z)$

$$\|zP(z)+h\|_4^4+\|zP(z)-h\|_4^4+\|zP(z)+0\|_4^4$$

$$=3\|P(z)\|_4^4+8h^2\|P(z)\|_2^2+2h^4.$$

Theorem *For $n \geq 0$, we have*

$$\beta_n(2)=\frac{2}{3}(n+1),$$

$$\beta_n(4)=\frac{8}{9}n^2+\frac{14}{9}n+\frac{2}{3}$$

and

$$\beta_n(6)=\frac{16}{9}n^3+4n^2+\frac{26}{9}n+\frac{2}{3}.$$

It is worth noting that the above technique can also be used to compute the averages of the norms of polynomials of height H. For example, one can show the following. Let $n \geq 0$ and $H \geq 1$ be integers and let

$$\mathcal{F}_n(H) := \left\{ \sum_{i=0}^{n} a_i z^i : |a_i| \leq H, a_i \in \mathbb{Z} \right\}$$

be the set of all the polynomials of height H and degree $\leq n$. Let

$$\beta_n(m, H) := \frac{1}{(2H+1)^{n+1}} \sum_{P \in \mathcal{F}_n(H)} \|P\|_m^m.$$

Theorem *For $n \geq 0$ and $H \geq 1$, we have*

$$\beta_n(2, H) = \frac{1}{3} H(H+1)(n+1),$$

$$\begin{aligned}\beta_n(4, H) = &\frac{2}{9} H^2(H+1)^2 n^2 \\ &+ \frac{1}{45} H(H+1)(19H^2 + 19H - 3)n \\ &\quad + \frac{1}{15} H(H+1)(3H^2 + 3H - 1)\end{aligned}$$

and

$$\beta_n(6,H) = \frac{2}{9}H^3(H+1)^3n^3 + \frac{1}{5}H^2(H+1)^2(3H^2+3H-1)n^2 + \frac{1}{315}H(H+1)(164H^4+328H^3+56H^2-108H+15)n + \frac{1}{21}H(H+1)(3H^4+6H^3-3H+1).$$

Derivative and reciprocal polynomials

If we replace z by z/w in the critical identities then we have homogeneous forms like

$$\begin{aligned}|z+hw|^4+|z-hw|^4 \\ &= 2(|z|^4+4h^2|z|^2|w|^2+h^4|w|^4 \\ &\qquad +h^2|w|^4(\left(\frac{z}{w}\right)^2+\left(\frac{\overline{z}}{\overline{w}}\right)^2)).\end{aligned}$$

Let $P^{(m)}(z)$ be the mth derivative of $P(z)$.

Theorem *For $n \geq 0$, we have, for $m \leq n$*

$$\frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_n} \|P^{(m)}\|_2^2$$

$$= m!^2 \sum_{l=m}^{n} \binom{l}{m}^2;$$

and

$$\frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_n} \|P^{(m)}\|_4^4$$

$$= 2m!^4 \left(\sum_{l=m}^{n} \binom{l}{m}^2 \right)^2 - m!^4 \sum_{l=m}^{n} \binom{l}{m}^4.$$

A polynomial $P(z)$ of degree n is **reciprocal** if $P(z) = P^*(z)$ where $P^*(z) = z^n P\left(\frac{1}{z}\right)$. Now

$$\|P(z) + z^{n+1}P^*(z)\|_4^4 + \|P(z) - z^{n+1}P^*(z)\|_4^4$$

$$= 12\|P\|_4^4.$$

This lets us prove that if n is odd the average $\|P\|_4^4$ over the reciprocal Littlewood polynomials in $\mathcal{L}_n$ is

$$3n^2 + 3n$$

if n is odd and

$$3n^2 + 3n + 1$$

if n is even.

The Complex Case

Lemma *For $m \geq l \geq 0$, we have*

$$\sum_{j=0}^{\min(l,m-l)} \binom{m}{2j}\binom{2j}{j}\binom{m-2j}{l-j} = \binom{m}{l}^2.$$

Lemma *Let $1 \leq m < k$ and $\zeta_k = e^{\frac{2\pi i}{k}}$. Then for any complex number z, we have*

$$\sum_{j=0}^{k-1} |z+\zeta_k^j|^{2m} = k\sum_{l=0}^{m} \binom{m}{l}^2 |z|^{2l}.$$

Let $n \geq 0$ and

$$\mathcal{L}_{n,k} := \left\{ \sum_{i=0}^{n} a_i z^i : a_i^k = 1 \right\}$$

be the set of all polynomials of degree $\leq n$ whose coefficients are kth root of unity.

Theorem *For $n \geq 0$ and $m < k$, we have*

$$\frac{1}{k^{n+1}} \sum_{P \in \mathcal{L}_{n,k}} \|P\|_{2m}^{2m}$$

$$= \sum_{l_1=0}^{m} \sum_{l_2=0}^{l_1} \cdots \sum_{l_n=0}^{l_{n-1}} \binom{m}{l_1}^2 \binom{l_1}{l_2}^2 \cdots \binom{l_{n-1}}{l_n}^2 .$$