MERIT FACTOR PROBLEMS

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Littlewood Type Problems

We are primarily concerned with polynomials with coefficients in the set $\{+1, -1\}$. Since many of these problems were raised by Littlewood we denote the set of such polynomials by $\mathcal{L}_n$ and refer to them as Littlewood polynomials. Specifically

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \;\; a_j \in \{-1, 1\} \right\}.$$

The following conjecture is due to Littlewood probably from some time in the fifties. It has been much studied and has associated with it a considerable literature

Conjecture. *It is possible to find $p_n \in \mathcal{L}_n$ so that*

$$C_1\sqrt{n+1} \le |p_n(z)| \le C_2\sqrt{n+1}$$

for all complex z of modulus 1. Here the constants C_1 and C_2 are independent of n.

Such polynomials are often called "locally flat". Because the L_2 norm of a polynomial from $\mathcal{L}_n$ is exactly $\sqrt{n+1}$ the constants must satisfy $C_1 \le 1$ and $C_2 \ge 1$.

It is still the case that no sequence is known that satisfies the lower bound.

A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials:

$$p_0(z) := 1, \qquad q_0(z) := 1$$

and

$$p_{n+1}(z) := p_n(z) + z^{2^n} q_n(z),$$

$$q_{n+1}(z) := p_n(z) - z^{2^n} q_n(z)$$

These have all coefficients ± 1 and are of degree $2^n - 1$. From

$$|p_{n+1}|^2 + |q_{n+1}|^2 = 2(|p_n|^2 + |q_n|^2)$$

we have for all z of modulus 1

$$|p_n(z)| \leq 2\sqrt{2}^n = \sqrt{2}\sqrt{\deg(p_n)}$$

and

$$|q_n(z)| \leq 2\sqrt{2}^n = \sqrt{2}\sqrt{\deg(q_n)}$$

This conjecture is complemented by a conjecture of Erdős.

Conjecture. *The constant C_2 in Littlewood's conjecture is bounded away from 1 (independently of n).*

This is also still open. Though a remarkable result of Kahane's shows that if the polynomials are allowed to have complex coefficients of modulus 1 then "locally flat" polynomials exist and indeed that it is possible to make C_1 and C_2 asymptotically arbitrarily close to 1.

Another striking result due to Beck proves that "locally flat" polynomials exist from the class of polynomials of degree n whose coefficients are 400th roots of unity.

Because of the monotonicity of the L_p norms it is relevant to rephrase Erdős' conjecture in other norms. Newman and Byrnes speculate that

$$||p||_4^4 \geq (6-\delta)n^2/5$$

for $p \in \mathcal{L}_n$ and n sufficiently large. This, of course, would imply Erdős' conjecture above. Here

$$||q||_p = \left(\int_0^{2\pi} |q(\theta)|^p \, d\theta/(2\pi)\right)^{1/p}$$

is the normalized p norm on the boundary of the unit disc.

It is possible to find a sequence of $p_n \in \mathcal{L}_n$ so that

$$||p_n||_4^4 \asymp (7/6)n^2.$$

This sequence is constructed out of the Fekete polynomials

$$f_p(z) := \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. One now takes the Fekete polynomials and cyclically permutes the coefficients by about $p/4$ to get the above example due to Turyn.

Problem. *Show for some absolute constant $\delta > 0$ and for all $p_n \in \mathcal{L}_n$*

$$||p||_4 \geq (1+\delta)\sqrt{n}$$

or even the much weaker

$$||p||_4 \geq \sqrt{n} + \delta.$$

A very interesting question is how to compute the minimal L_4 Littlewood polynomials (say up to degree 200).

A Barker polynomial

$$p(z) := \sum_{k=0}^{n} a_k z^k$$

with each $a_k \in \{-1, +1\}$ so that

$$p(z)\overline{p(z)} := \sum_{k=-n}^{n} c_k z^k$$

satisfies $c_0 = n + 1$ and

$$|c_j| \leq 1, \qquad j = 1, 2, 3 \ldots .$$

Here

$$c_j = \sum_{k=0}^{n-j} a_k a_{n-k} \qquad and \qquad c_{-j} = c_j .$$

If $p(z)$ is a Barker polynomial of degree n then

$$||p||_4 \leq ((n+1)^2 + 2n))^{1/4}$$

The nonexistence of Barker polynomials of degree n is now shown by showing

$$||p_n||_4 \geq (n+1)^{1/2} + (n+1)^{-1/2}/2.$$

This is even weaker than the weak form of the preceding problem.

It is conjectured that no Barker polynomials exist for $n > 12$.

We can compute the expected L_p norm of Littlewood polynomials (B and Lockhart).

For random $q_n \in \mathcal{L}_n$

$$\frac{\mathrm{E}(||q_n||_p)}{n^{1/2}} \to (\Gamma(1+p/2))^{1/p}$$

and for derivatives

$$\frac{\mathrm{E}(||q_n^{(r)}||_p)}{n^{(2r+1)/2}} \to (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

Explicit Merit Calculations

Our main purpose is to give explicit formulas for the L_4 norms (on the boundary of the unit disc) and hence, also the merit factors of various polynomials that are closely related to the Fekete polynomials.

As usual the L_α norm on the boundary of the unit disc is defined by

$$\|p\|_\alpha = \left(\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\alpha \, d\theta \right)^{1/\alpha}.$$

The L_4 norm of a polynomial is particularly easy to work with because it can be computed as the square root of the L_2 norm of $p(z)\overline{p(z)}$ and hence, computes exactly as the fourth root of the sum of the squares of the coefficients of $p(z)\overline{p(z)}$. In contrast, the supremum norm or other L_p norms, where p is not an even integer, are computationally difficult.

Let q be a prime number and let $\left(\frac{\cdot}{q}\right)$ be the Legendre symbol.

The Fekete polynomials are defined by

$$f_q(z) := \sum_{k=1}^{q-1} \left(\frac{k}{q}\right) z^k$$

and the closely related polynomials

$$F_q(z) := 1 + f_q(z) = 1 + \sum_{k=1}^{q-1} \left(\frac{k}{q}\right) z^k.$$

The half–Fekete polynomials are defined by

$$G_q(z) := \sum_{k=1}^{(q-1)/2} \left(\frac{k}{q}\right) z^k.$$

If we cyclically permute the coefficients of f_q by about $q/4$ places we get an example of Turyn's which we denote by

$$R_q(z) := \sum_{k=0}^{q-1} \left(\frac{k + [q/4]}{q} \right) z^k$$

where $[\cdot]$ denotes the nearest integer, and we denote the general shifted Fekete polynomials by

$$f_q^t(z) := \sum_{k=0}^{q-1} \left(\frac{k+t}{q} \right) z^k.$$

Note that R has one coefficient that is zero (from the permutation of the constant term in f). For example

$$f_{11} := -x^{10} + x^9 - x^8 - x^7 - x^6 + x^5 + x^4 + x^3 - x^2 + x$$

and

$$R_{11} := -x^{10} + x^9 - x^7 + x^6 - x^5 - x^4 - x^3 + x^2 + x + 1.$$

The explicit formulas involve the class number of the imaginary quadratic field of $\mathbb{Q}(\sqrt{-d})$ which is denoted by $h(-d)$. For any odd prime d it can be computed as

$$h(-d) = \lambda_d \sum_{k=1}^{(d-1)/2} \left(\frac{k}{d}\right)(-1)^k = \lambda_d G_d(-1)$$

where

$$\lambda_d := \begin{cases} 1 & \text{if} \quad d \equiv 1, 7 \pmod 8, \\ -1/3 & \text{if} \quad d \equiv 3 \pmod 8, \\ -1 & \text{if} \quad d \equiv 5 \pmod 8. \end{cases}$$

For primes $d \equiv 3 \pmod 4$ it can also be computed as

$$h(-d) = -\frac{f_d'(1)}{d} = -\frac{1}{d} \sum_{k=1}^{d-1} \left(\frac{k}{d}\right) k$$

(this sum is 0 for $d \equiv 1 \pmod 4$).

There are two natural measures of smallness for the L_4 norm of a polynomial p. One is the ratio of the L_4 norm to the L_2 norm, $\|p\|_4/\|p\|_2$. The other (equivalent) measure is the merit factor, defined by

$$\mathrm{MF}(p) = \frac{\|p\|_2^4}{\|p\|_4^4 - \|p\|_2^4}.$$

Littlewood polynomials are the set

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^{n} a_j x^j \,, \ \ a_j \in \{-1, 1\} \right\}.$$

The L_2 norm of any element of $\mathcal{L}_{n-1}$ is $\sqrt{n}$ and this is, of course, a lower bound for the L_4 norm.

The expected L_4 norm of an element of $\mathcal{L}_n$ is is $2^{1/4}\sqrt{n}$. The expected merit factor is thus 1.

The $\{R_q\}$ above are a sequence with asymptotic merit factor 6. Golay gives a heuristic argument for this observation of Turyn's and this is proved rigorously by T. Høholdt and H. Jensen

The Fekete polynomials themselves have asymptotic merit factor 3/2 and different amounts of cyclic permutations can give rise to any asymptotic merit factor between 3/2 and 6.

Golay speculates that 6 may be the largest possible asymptotic merit factor. He writes "the eventuality must be considered that no systematic synthesis will ever be found which will yield higher merit factors."

Newman and Byrnes, apparently independently, make a similar conjecture. As do Høholdt and Jensen.

Computations by a number of people on polynomials up to degree 200 are equivocal. See the web page of A. Reinholz at http://borneo.gmd.de/~andy/ACR.html.

The Fekete polynomial f_q has modulus $\sqrt{q}$ at each qth root of unity (as does f_q^t) and one might hope that they also satisfy the upper bound in Littlewood's conjecture but Montgomery shows that this is not the case.

Littlewood's conjecture is that it is possible to find $p_n \in \mathcal{L}_{n-1}$ so that

$$C_1\sqrt{n} \leq |p_n(z)| \leq C_2\sqrt{n}$$

for all z of modulus 1 and for two constants C_1, C_2 independent of n.

2. Results

Theorem 1. *For q an odd prime, the Fekete polynomial,*

$$f_q(z) := \sum_{k=1}^{q-1} \left(\frac{k}{q}\right) z^k$$

satisfies

$$\|f_q\|_4^4 = \frac{5q^2}{3} - 3q + \frac{4}{3} - \gamma_q$$

where

$$\gamma_q := \begin{cases} 0 & \textit{if} \quad q \equiv 1 \pmod 4, \\ 12(h(-q))^2 & \textit{if} \quad q \equiv 3 \pmod 4. \end{cases}$$

Theorem 2. *For q an odd prime, the modified Fekete polynomial,*

$$F_q(z) := 1 + \sum_{k=1}^{q-1} \left(\frac{k}{q}\right) z^k$$

satisfies

$$\|F_q\|_4^4 = \frac{5q^2}{3} + q - \frac{5}{3} - \gamma_q$$

where

$$\gamma_q := \begin{cases} 0 & \textit{if} \quad q \equiv 1 \pmod 4, \\ 12h(-q)(h(-q)+1) & \textit{if} \quad q \equiv 3 \pmod 4. \end{cases}$$

Theorem 3. *For q an odd prime the half–Fekete polynomials*

$$G_q(z) := \sum_{k=1}^{(q-1)/2} \left(\frac{k}{q}\right) z^k.$$

satisfy

$$\|G_q\|_4^4 = \frac{q^2}{3} - \frac{q}{2} + \frac{1}{6} - \gamma_q (h(-q))^2$$

where

$$\gamma_q := \begin{cases} 0 & \textit{if} \quad q \equiv 1 \pmod 4, \\ 2 & \textit{if} \quad q \equiv 7 \pmod 8, \\ 6 & \textit{if} \quad q \equiv 3 \pmod 8. \end{cases}$$

The exact same formulae above hold for the polynomials $(f_q(z) + f_q(-z))/2$ and $(f_q(z) - f_q(-z))/2$.

Theorem 4. *For q an odd prime, the Turyn type polynomials*

$$R_q(z) := \sum_{k=0}^{q-1} \left(\frac{k + [q/4]}{q} \right) z^k$$

where $[\cdot]$ denotes the nearest integer, satisfy

$$\|R_q\|_4^4 = \frac{7q^2}{6} - q - \frac{1}{6} - \gamma_q$$

and

$$\gamma_q := \begin{cases} h(-q)(h(-q)-4) & \text{if} \quad q \equiv 1, 5 \pmod 8, \\ 12(h(-q))^2 & \text{if} \quad q \equiv 3 \pmod 8, \\ 0 & \text{if} \quad q \equiv 7 \pmod 8. \end{cases}$$

Theorem 5. *For q an odd prime, the shifted Fekete polynomials*

$$f_q^t(z) := \sum_{k=0}^{q-1} \left(\frac{k+t}{q}\right) z^k$$

satisfy

$$\|f_q^t\|_4^4 =$$

$$\frac{1}{3}(5q^2+3q+4)+8t^2-4qt-8t-\frac{8}{q^2}\left(1-\frac{1}{2}\left(\frac{-1}{q}\right)\right)\left|\sum_{n=1}^{q-1} n\left(\frac{n+t}{q}\right)\right|^2,$$

and

$$\|f_q^{q-t+1}\|_4^4 = \|f_q^t\|_4^4$$

if $1 \le t \le (q-1)/2$.

Montgomery shows that the maximum modulus of $f_q(z)$ at the $2q$th root of unity is at least $\frac{2}{\pi}\sqrt{q}\log\log q$.

Corollary 6. *For q an odd prime, we have*

$$\sum_{j=0}^{q-1} |f_q(-e^{\frac{2\pi i k}{q}})|^4 = \frac{q}{3}(7q-8)(q-1) - 2q\gamma_q$$

where γ_q is the same as in Theorem 1.

Theorem 7. *For q an odd prime, the shifted Fekete polynomials*

$$f_q^t(z) := \sum_{k=0}^{q-1} \left(\frac{k+t}{q}\right) z^k$$

satisfy

$$\|f_q^t\|_4^4 = \|f_q^{q-t+1}\|_4^4 = \frac{5q^2}{3} + 8t^2 - 4qt + O(q(\log q)^2)$$

if $1 \leq t \leq (q-1)/2$.

Theorem 7 follows from Theorem 5 on observing that

$$\frac{1}{q}\sum_{n=1}^{q-1} n\left(\frac{n+k}{q}\right) = \sum_{n=1}^{k-1}\left(\frac{n}{q}\right) + \frac{1}{q}\sum_{n=1}^{q-1} n\left(\frac{n}{q}\right).$$

This is coupled with the known estimate

$$\left|\sum_{n=1}^{k-1}\left(\frac{n}{q}\right)\right| < q^{\frac{1}{2}} \log q$$

and the observation that

$$-\frac{1}{q}\sum_{n=1}^{q-1} n\left(\frac{n}{q}\right)$$

equals the class number, $h(-q)$, for primes $q \equiv 3 \pmod 4$ and is zero for primes $q \equiv 1 \pmod 4$. The asymptotics of Turyn et al mentioned previously are the above theorem in the case where t is a constant multiple of q.

Theorem 8. *Let*

$$L_n(z) := \sum_{k=0}^{n-1} e^{\frac{k(k+1)\pi i}{n}} z^k$$

$$\|L_n\|_4^4 = n^2 + \frac{2n^{3/2}}{\pi} + \delta_n \frac{n^{1/2}}{3} + O(n^{-1/2})$$

where

$$\delta_n := \begin{cases} -2 & \textit{if} \quad n \equiv 0, 1 \pmod 4, \\ 1 & \textit{if} \quad n \equiv 2, 3 \pmod 4. \end{cases}$$

The above example of Littlewoods depends on the asymptotic series for

$$\sum_{j=1}^{n-1} \frac{sin^2(j^2\pi/n)}{sin^2(j\pi/n)}$$

because, in the above notation,

$$\|L_n\|_4^4 = n^2 + 2\sum_{j=1}^{n-1} \frac{sin^2(j^2\pi/n)}{sin^2(j\pi/n)}.$$

Let q be a prime and χ be a non-principal character mod q. Let

$$f_\chi^t(z) := \sum_{n=0}^{q-1} \chi(n+t) z^n$$

for $1 \le t \le q$ be the character polynomial associated to χ (cyclically permuted t places).

Theorem 9. *For any non-principal and non-real character χ modulo q and $1 \le t \le q$, we have*

$$\|f_\chi^t(z)\|_4^4 = \frac{4}{3} q^2 + O(q^{3/2} \log^2 q)$$

where the implicit constant is independent of t and q. Here $\| \cdot \|_4$ denotes the L_4 norm on the unit circle.

It follows from this that all cyclically permuted character polynomials associated with non-principal and non-real characters have merit factors that approach 3.

We also compute the averages of the L_4 norms:

Theorem 10. *Let q be a prime number. We have*

$$\sum_{\chi \pmod q} \|f_\chi^t\|_4^4 = (2q-3)(q-1)^2$$

where the summation is over all characters modulo q.

Proof of Theorem 5

Let q be a prime number and, as before, let

$$f_q(z) := \sum_{n=1}^{q-1} \left(\frac{n}{q}\right) z^n$$

be the Fekete polynomial. Define

$$\epsilon_q := \begin{cases} 1 & \text{if} \quad q \equiv 1 \pmod 4, \\ i & \text{if} \quad q \equiv 3 \pmod 4. \end{cases}$$

Then we have the following well-known result

$$f_q(\omega^k) = \epsilon_q \sqrt{q} \left(\frac{k}{q}\right).$$

for $k = 0, 1, \cdots, q-1$, where $\omega := e^{2\pi i/q}$.

Lemma 1. *For any $1 \le t \le q$, we have*

$$\sum_{b=1}^{q-1} \left(\frac{b}{q}\right) \frac{\omega^{bt}}{\omega^b - 1} = \frac{\epsilon_q}{\sqrt{q}} \sum_{n=1}^{q-1} n \left(\frac{n+t}{q}\right).$$

Lemma 2. *If* $1 \le k \le q$, *then*

$$\sum_{\substack{n,m=1 \\ k+n+m\equiv 0 \pmod q}}^{q-1} nm = \frac{q}{6}(q^2 - 6q - 1 + 6k + 3qk - 3k^2).$$

Lemma 3. *If* $1 \le k \le q$, *then*

$$\sum_{\substack{a,b=1 \\ a\neq b}}^{q-1} \frac{\omega^{(a-b)k}}{(\omega^{a-b}-1)^2} = -\frac{1}{12}(q-2)(q^2+6q+5-12k-6qk+6k^2).$$

As before let $f_q^t(z)$ be the shifted Fekete polynomial obtained by shifting the coefficients to the left by t where $1 \le t \le q$. So $f_q^q(z) = f_q(z)$. Then we have

$$f_q^t(\omega^k) = \omega^{-tk} f_q(\omega^k)$$

for any $0 \le k \le q-1$. We are going to evaluate the following summation

$$\sum_{k=0}^{q-1} |f_q^t(-\omega^k)|^4.$$

We use the basic approach of T. Høholdt and H. Jensen which is by interpolation at the $2q$th roots of unity. Using the Lagrange interpolation formula at the qth roots of unity, we have

$$f_q^t(z) = \frac{1}{q}\sum_{j=0}^{q-1}\frac{z^q-1}{z-\omega^j}\omega^j f_q^t(\omega^j).$$

It follows that

$$\begin{aligned}\sum_{k=0}^{q-1}|f_q^t(-\omega^k)|^4 &= \frac{16}{q^4}\sum_{k=0}^{q-1}\left|\sum_{j=0}^{q-1}\frac{\omega^j}{\omega^k+\omega^j}f_q^t(\omega^j)\right|^4\\ &= \frac{16}{q^4}\sum_{a,b,c,d=0}^{q-1} f_q^t(\omega^a)\bar{f}_q^t(\omega^b)f_q^t(\omega^c)\bar{f}_q^t(\omega^d)\omega^{a+c}\\ &\qquad\times\sum_{k=0}^{q-1}\frac{1}{\omega^k+\omega^a}\frac{\omega^k}{\omega^k+\omega^b}\frac{1}{\omega^k+\omega^c}\frac{\omega^k}{\omega^k+\omega^d}.\end{aligned}$$

We then group the terms in the above summation over a, b, c and d by the following cases:

(1) $a = c$ and $a \neq b \neq d$,
(2) $a = b = c \neq d$,
(3) $a = b \neq c = d$,
(4) $a = b = c = d$,
(5) $a \neq b \neq c \neq d$,

and we obtain the following formula

$$\sum_{k=0}^{q-1} |f_q^t(-\omega^k)|^4 = \frac{16}{q^4}(A+B+C+D)$$

where

$$A = \frac{1}{48}q^2(q^2+2)\sum_{a=0}^{q-1}|f_q^t(\omega^a)|^4$$

$B =$

$$\frac{q^2}{4}\sum_{\substack{a,b=0\\a\neq b}}^{q-1}|f_q^t(\omega^a)|^2(\bar{f}_q^t(\omega^a)f_q^t(\omega^b)\omega^b + f_q^t(\omega^a)\bar{f}_q^t(\omega^b)\omega^a)\left(\frac{\omega^a+\omega^b}{(\omega^a-\omega^b)^2}\right)$$

$$C = -\frac{q^2}{4}\sum_{\substack{a,b,c=0\\a\neq b\neq c}}^{q-1} 2|f_q^t(\omega^a)|^2\left(\frac{f_q^t(\omega^b)\bar{f}_q^t(\omega^c)\omega^{a+b} + f_q^t(\omega^c)\bar{f}_q^t(\omega^b)\omega^{a+c}}{(\omega^b-\omega^a)(\omega^c-\omega^a)}\right)$$

$$-\frac{q^2}{4}\sum_{\substack{a,b,c=0\\a\neq b\neq c}}^{q-1}\frac{f_q^t(\omega^a)^2\bar{f}_q^t(\omega^b)\bar{f}_q^t(\omega^c)\omega^{2a} + \bar{f}_q^t(\omega^a)^2 f_q^t(\omega^b)f_q^t(\omega^c)\omega^{b+c}}{(\omega^b-\omega^a)(\omega^c-\omega^a)}$$

$$D = -\frac{q^2}{4}$$

$$\sum_{\substack{a,b=0\\a\neq b}}^{q-1}\frac{4|f_q^t(\omega^a)|^2|f_q^t(\omega^b)|^2\omega^{a+b} + f_q^t(\omega^b)^2\bar{f}_q^t(\omega^a)^2\omega^{2b} + f_q^t(\omega^a)^2\bar{f}_q^t(\omega^b)^2\omega^{2a}}{(\omega^a-\omega^b)^2}.$$

Here A, B, C and D are the sum of terms according to the above cases (1), (2), (3) and (4) respectively and the sum of terms corresponding to the case (5) is zero.

We now evaluate A, B, C and D separately. The details are formidable.

To prove Theorem 4 we need .

Lemma 4. *Let q be a prime and $q > 3$. Then we have*

$$\sum_{n=1}^{[\frac{q}{4}]-1} \left(\frac{n}{q}\right) = \begin{cases} \frac{1}{2}h(-q) - 1 & \text{if} \quad q \equiv 1 \pmod 4, \\ 0 & \text{if} \quad q \equiv 3 \pmod 8, \\ h(-q) & \text{if} \quad q \equiv 7 \pmod 8. \end{cases}$$

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