The Amazing Number π

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Abstract: A version of this address was given at the celebration to replace the lost tombstone of Ludolph van Ceulen at the Pieterskerk (St Peter's Church) in Leiden on the fifth of July, 2000.

It honours the particular achievements of Ludolph as well as the long and important tradition of intellectual inquiry associated with understanding the number π and numbers generally.

The amazing number π:

The history of π parallels virtually the entire history of Mathematics. At times it has been of central interest and at times the interest has been quite peripheral.

Certainly Lindemann's proof of the transcendence of π was one of highlights of nineteenth century mathematics and stands as one of the seminal achievements of the millennium

Ueber die Zahl π.*)

Von

F. Lindemann in Freiburg i. Br.

Bei der Vergeblichkeit der so ausserordentlich zahlreichen Versuche**), die Quadratur des Kreises mit Cirkel und Lineal auszuführen, hält man allgemein die Lösung der bezeichneten Aufgabe für unmöglich; es fehlte aber bisher ein Beweis dieser Unmöglichkeit; nur die Irrationalität von π und von π^2 ist festgestellt. Jede mit Cirkel und Lineal ausführbare Construction lässt sich mittelst algebraischer Einkleidung zurückführen auf die Lösung von linearen und quadratischen Gleichungen, also auch auf die Lösung einer Reihe von quadratischen Gleichungen, deren erste rationale Zahlen zu Coefficienten hat, während die Coefficienten jeder folgenden nur solche irrationale Zahlen enthalten, die durch Auflösung der vorhergehenden Gleichungen eingeführt sind. Die Schlussgleichung wird also durch wiederholtes Quadriren übergeführt werden können in eine Gleichung geraden Grades, deren Coefficienten rationale Zahlen sind. Man wird sonach die Unmöglichkeit der Quadratur des Kreises darthun, wenn man nachweist, dass *die Zahl π überhaupt nicht Wurzel einer algebraischen Gleichung irgend welchen Grades mit rationalen Coefficienten sein kann.* Den dafür nöthigen Beweis zu erbringen, ist im Folgenden versucht worden.

Die wesentliche Grundlage der Untersuchung bilden die Relationen zwischen gewissen bestimmten Integralen, welche Herr Hermite angewandt hat***), um den transcendenten Charakter der Zahl e festzustellen. In § 1 sind deshalb die betreffenden Formeln zusammengestellt; § 2 und § 3 geben die Anwendung dieser Formeln zum Beweise des erwähnten Satzes; § 4 enthält weitere Verallgemeinerungen.

*) Vergl. eine Mittheilung des Hrn. Weierstrass an die Berliner Akademie, vom 22. Juni 1882.

**) Man sehe die Artikel *Cyclometrie*, *Quadratur* und *Rectification* in Klügel's mathematischen Wörterbuche.

***) Sur la fonction exponentielle, Paris 1874 (auch Comptes rendus, t. LXXVII, 1873).

One of the low points was the Indiana State legislatures attempts to legislate a value of π in 1897. An attempt as plausible as repealing the law of gravity.

Bill No. 246, 1897. State of Indiana.

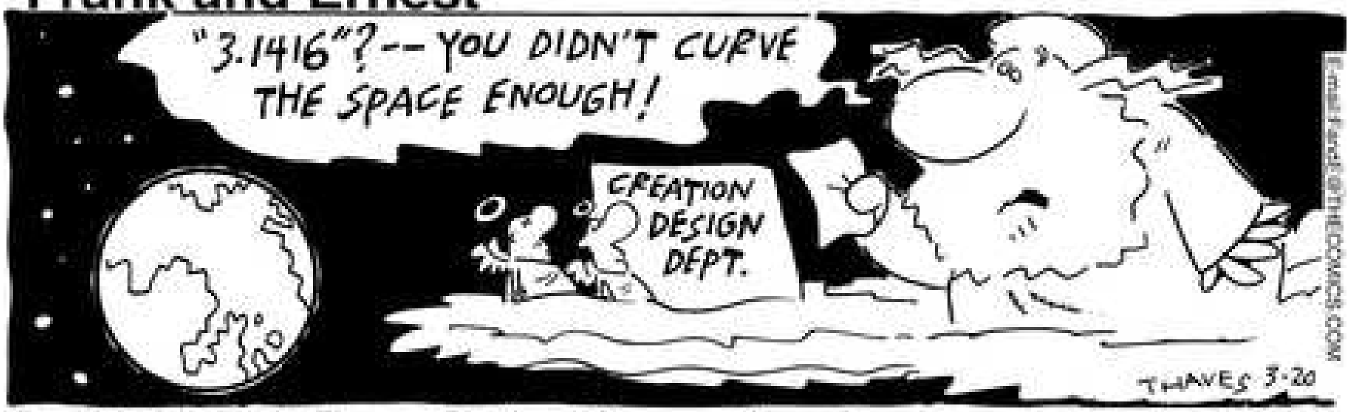
“Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side.”

The bill was viewed as having financial value:

"The case is perfectly simple. If we pass this bill which establishes a new and correct value of π, the author offers our state without cost the use of this discovery and its free publication in our school textbooks, while everyone else must pay him a royalty."

By chance Professor C.A. Waldo of Purdue was in the Senate for a reading of the bill. He convinced Senators that the bill was nonsense and it was tabled. (Presumably it is still tabled.)

Frank and Ernest



The amount of human ingenuity that has gone into understanding the nature of π and computing its digits is quite phenomenal and begs the question "why π?".

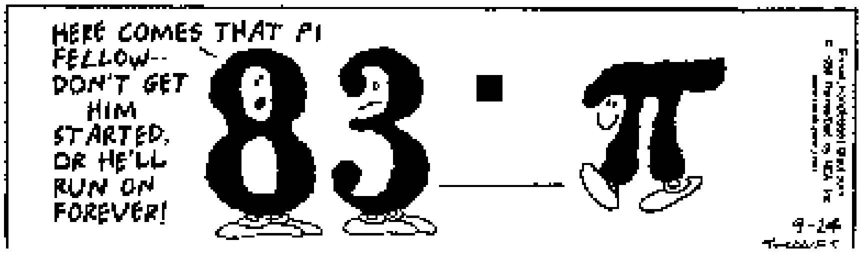
What is π?

Geometrically, of course, π is the circumference of a circle of diameter 1.

Ludolph had computed by his death in 1610

$\pi = 3.14159265358979323$
$846264338327950 88\ldots$

FRANK AND ERNEST® by Bob Thaves



The Greek notion of number was quite different from ours. The Greek notion was based on length and numbers that existed were numbers that could be drawn (with just an unmarked ruler and compass).

Is π a number (in the Greek sense)?

Lindemann's proof of the transcendence of π in 1882 settles the issue that π is not constructible by the Greek rules.

Ueber die Transcendenz der Zahlen e und π.*)

Von

DAVID HILBERT in Königsberg i. Pr.

Man nehme an, die Zahl e genüge der Gleichung n^{ten} Grades

$$a + a_1 e + a_2 e^2 + \cdots + a_n e^n = 0,$$

deren Coefficienten $a, a_1, \ldots, a_n$ ganze rationale Zahlen sind. Wird die linke Seite dieser Gleichung mit dem Integral

$$\int_0^\infty = \int_0^\infty z^\varrho [(z-1)(z-2)\cdots(z-n)]^{\varrho+1} e^{-z} dz$$

multiplicirt, wo ϱ eine ganze positive Zahl bedeutet, so entsteht der Ausdruck

$$a\int_0^\infty + a_1 e \int_0^\infty + a_2 e^2 \int_0^\infty + \cdots + a_n e^n \int_0^\infty$$

und dieser Ausdruck zerlegt sich in die Summe der beiden folgenden Ausdrücke:

$$P_1 = a\int_0^\infty + a_1 e\int_1^\infty + a_2 e^2 \int_2^\infty + \cdots + a_n e^n \int_n^\infty,$$

$$P_2 = \qquad a_1 e \int_0^1 + a_2 e^2 \int_0^2 + \cdots + a_n e^n \int_0^n.$$

Die Formel

$$\int_0^\infty z^\varrho e^{-z} dz = \varrho\,!$$

zeigt, dass das Integral $\int_0^\infty$ eine ganze rationale durch $\varrho\,!$ theilbare Zahl ist und ebenso leicht folgt, wenn man bezüglich die Substitutionen $z = z' + 1,\ z = z' + 2, \ldots, z = z' + n$ anwendet, dass

$$e\int_1^\infty,\ e^2\int_2^\infty, \ldots, e^n \int_n^\infty$$

ganze rationale durch $(\varrho+1)!$ theilbare Zahlen sind. Daher ist auch

*) Abdruck aus Nr. 2 der Göttinger Nachrichten v. J. 1893.

Does this tell us everything we wish to know about π. No, our ignorance is still much more profound than our knowledge!

What about $\pi + e$? This embarrassingly easy question is currently totally intractable.

Francois Viète

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

(ca 1579)

William Brouncker

$$\pi = \frac{4}{1+\frac{1}{2+\frac{9}{2+\frac{25}{2+\cdots}}}}$$

(ca 1650)

James Gregory

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(1450–1671)

Srinivasa Ramanujan

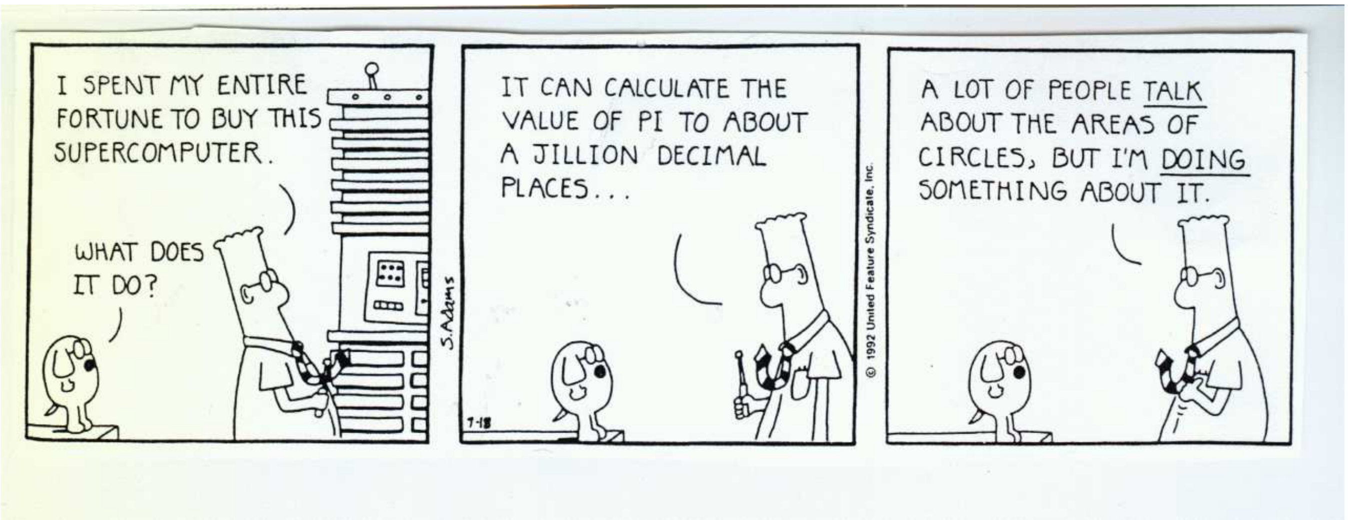
$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}.$$

(1914)

Why compute the digits of π?

Sometimes it is necessary to do so, though hardly ever more than six or so digits are ever really needed.

Whatever the personal motivations π has been much computed and a surprising amount has been learnt along the way.


I SPENT MY ENTIRE FORTUNE TO BUY THIS SUPERCOMPUTER.
WHAT DOES IT DO?
S. ADAMS
7-18
IT CAN CALCULATE THE VALUE OF PI TO ABOUT A JILLION DECIMAL PLACES...
© 1992 United Feature Syndicate, Inc.
A LOT OF PEOPLE TALK ABOUT THE AREAS OF CIRCLES, BUT I'M DOING SOMETHING ABOUT IT.

Bible	-550?	1
Archimedes	-250?	7?
Al-Kashi	1429	14
Van Ceulen	1610	35
Dase	1844	200
Shanks	1874	707 (527)
Ferguson et al	1946	620
ENIAC et al	1949	2,037
Genuys	1958	10,000
Shanks	1961	100,265
Guilloud	1973	1,001,250
Kanada et al	1982	16,777,206
Kanada et al	1987	134,217,700
Chudnovskys	1989	1 Billion
Kanada	1997	51 Billion
Kanada	1999	206 Billion
Kanada	2005	1 Trillion

In constructing the all star hockey team of great mathematicians, there seems to be pretty wide agreement that the front line consists of Archimedes, Newton, and Gauss.

Both Archimedes and Newton invented methods for computing π. In Newton's case this was an application of his newly invented calculus.

I might add Hilbert and Euler next (on defense) and both of these mathematicians also contribute to the story of π.

Perhaps von Neumann is in goal.

$\overline{\text{II}}$ degree $\quad \sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} = 1$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} - \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{2}{1+p} = \frac{2}{m}.$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} - \sqrt[3]{\frac{\beta^2}{\alpha}} = m.$$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{4}{m}.$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} + \sqrt[3]{\frac{\beta^2}{\alpha}} = m^2.$$

$\overline{\text{V}}$ degree $\quad \sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$

$\overline{\text{XI}}$ degree $\quad \sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 6\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}$

$$+ 3\sqrt{3}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right\} = 1$$

~~III degree~~ $\cancel{\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}} \pm \sqrt[4]{\frac{\beta^3}{\alpha}} -}$

$\overline{\text{IX}}$ degree $\quad m = 3\cdot\dfrac{1+2\sqrt[3]{\beta}}{1+2\sqrt[3]{1-\alpha}};$

$\overline{\text{IV}}$ degree $\quad m = \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{4}{m}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$\overline{\text{VII}}$ degree $\quad m = \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{7}{m}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 3\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$\overline{\text{I}}, \overline{\text{II}}, \overline{\text{IV}}$ and $\overline{\text{VIII}}$.

$$\frac{1 - \sqrt[3]{\alpha\delta} - \sqrt[3]{(1-\alpha)(1-\delta)}}{3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)}} = \frac{1+\frac{1\cdot2}{3^2}\beta+\&c}{1+\frac{1\cdot2}{3^2}\alpha+\&c}\cdot\frac{1+\frac{1\cdot2}{3^2}\gamma+\&c}{1+\frac{1\cdot2}{3^2}\delta+\&c}$$

$\overline{\text{I}}, \overline{\text{II}}, \overline{\text{VII}}, \overline{\text{XIV}}$ or $\overline{\text{I}}, \overline{\text{IV}}, \overline{\text{V}}, \overline{\text{XX}}$.

$$\frac{1+2\left(\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)}\right)}{1+2\left(\sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)}\right)} = \frac{1+\frac{1\cdot2}{3^2}\beta+\&c}{1+\frac{1\cdot2}{3^2}\alpha+\&c}\cdot\frac{1+\frac{1\cdot2}{3^2}\gamma+\&c}{1+\frac{1\cdot2}{3^2}\delta+\&c}$$

One doesn't often think of a problem like this having economic benefits. But as is often the case with pure mathematics and curiosity driven research the rewards can be surprising.

Large recent records depend on three things.

Better algorithms for π.

Larger and faster computers.

An understanding of how to do arithmetic with numbers that are billions of digits long.

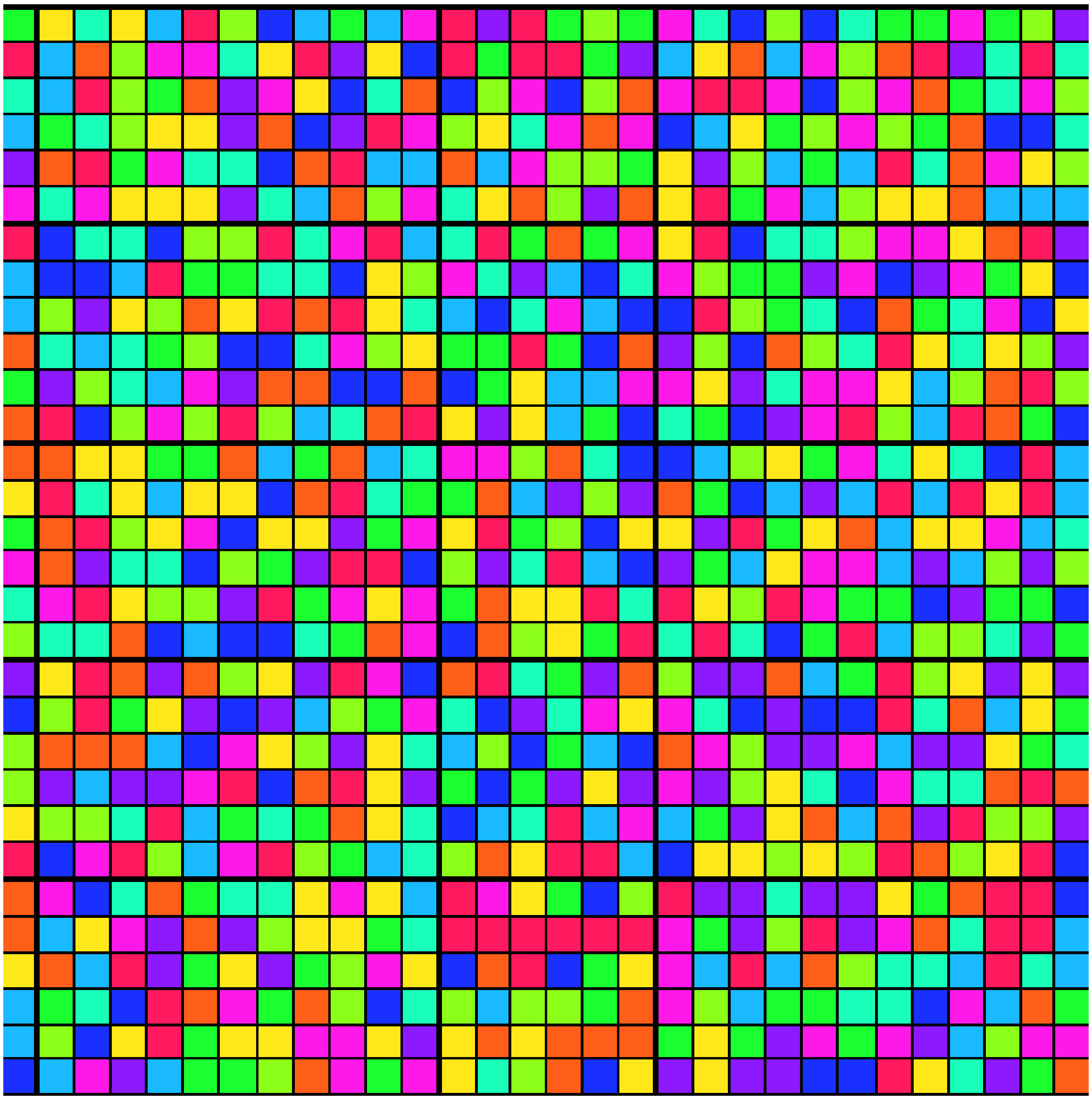
Understanding arithmetic is an interesting and illuminating story in its own right. A hundred years ago we knew how to add and multiply – do it the way we all learned in school. Now we are not so certain.

The mathematical technology that allows for multiplying very large numbers together is essentially the same as the mathematical technology that allows image processing devices like CAT scanners to work (FFTs).

Most recent records are set when new computers are being installed and tested. (Recent records are more or less how many digits can be computed in a day – a reasonable amount of test time on a costly machine.)

The computation of π seems to stretch the machine and there is a history of uncovering subtle and sometimes not so subtle bugs at this stage.

What do the calculations of π reveal and what does one expect? One expects that the digits of π should look random – that roughly one out of each ten digits should be a 7 etc. This appears to be true at least for the first few hundred billion.



The question of whether there are subtle patterns in the digits is an interesting one.

Looking for subtle patterns in long numbers is exactly the kind of problem one needs to tackle in handling the human genome (a chromosome is just a large number in base 4, at least to a mathematician).