The Unreasonable Efficacy of Symbolic Computation

—or—

Imagine if Gauss had Maple.

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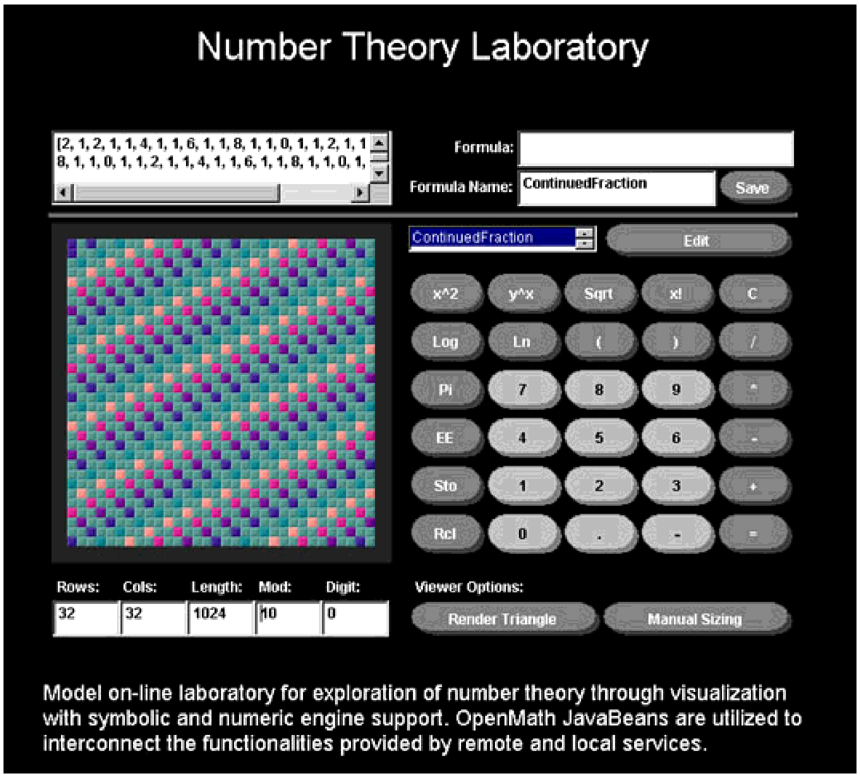
Peter Borwein's Homepage



Number Theory Laboratory

[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 0, 1, 1, 2, 1, 1
8, 1, 1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 0, 1,

Formula:

Formula Name: ContinuedFraction

ContinuedFraction

Rows: 32
Cols: 32
Length: 1024
Mod: 10
Digit: 0

Model on-line laboratory for exploration of number theory through visualization with symbolic and numeric engine support. OpenMath JavaBeans are utilized to interconnect the functionalities provided by remote and local services.

Abstract: Modern symbolic computation packages such as Maple have been far more successful than even their inventors could have imagined twenty years ago.

They play a critical role in how large parts of modern mathematics grow and evolve and can be anticipated to play an even larger role as they become increasingly more sophisticated.

It is not too grandiose to talk about a paradigm shift in mathematics with such tools at its core.

Symbolic algebra packages (Maple and Mathematica) have over the last fifteen years reached a remarkable degree of sophistication.

Rather difficult problems, like exact integration of elementary functions have been significantly attacked.

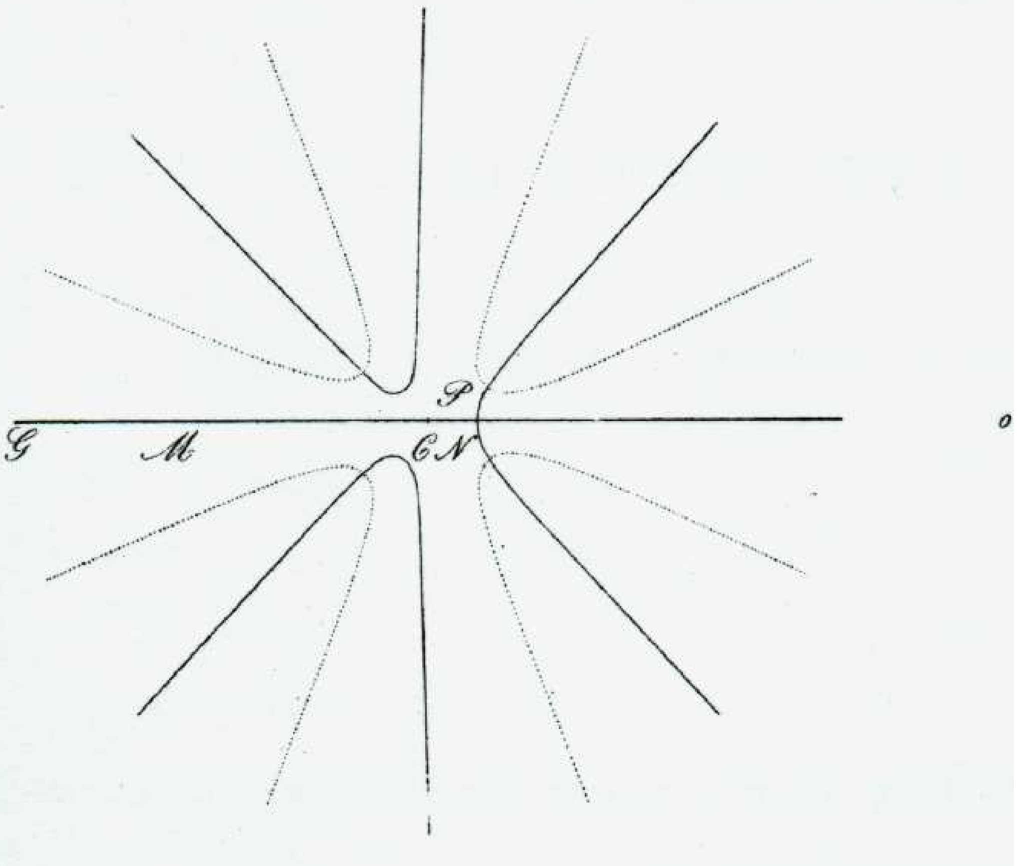
A number of the most important algorithms of the twentieth century like FFT's and LLL are centrally incorporated.

The great success of the symbolic algebra packages has been their mathematical generality and ease of use.

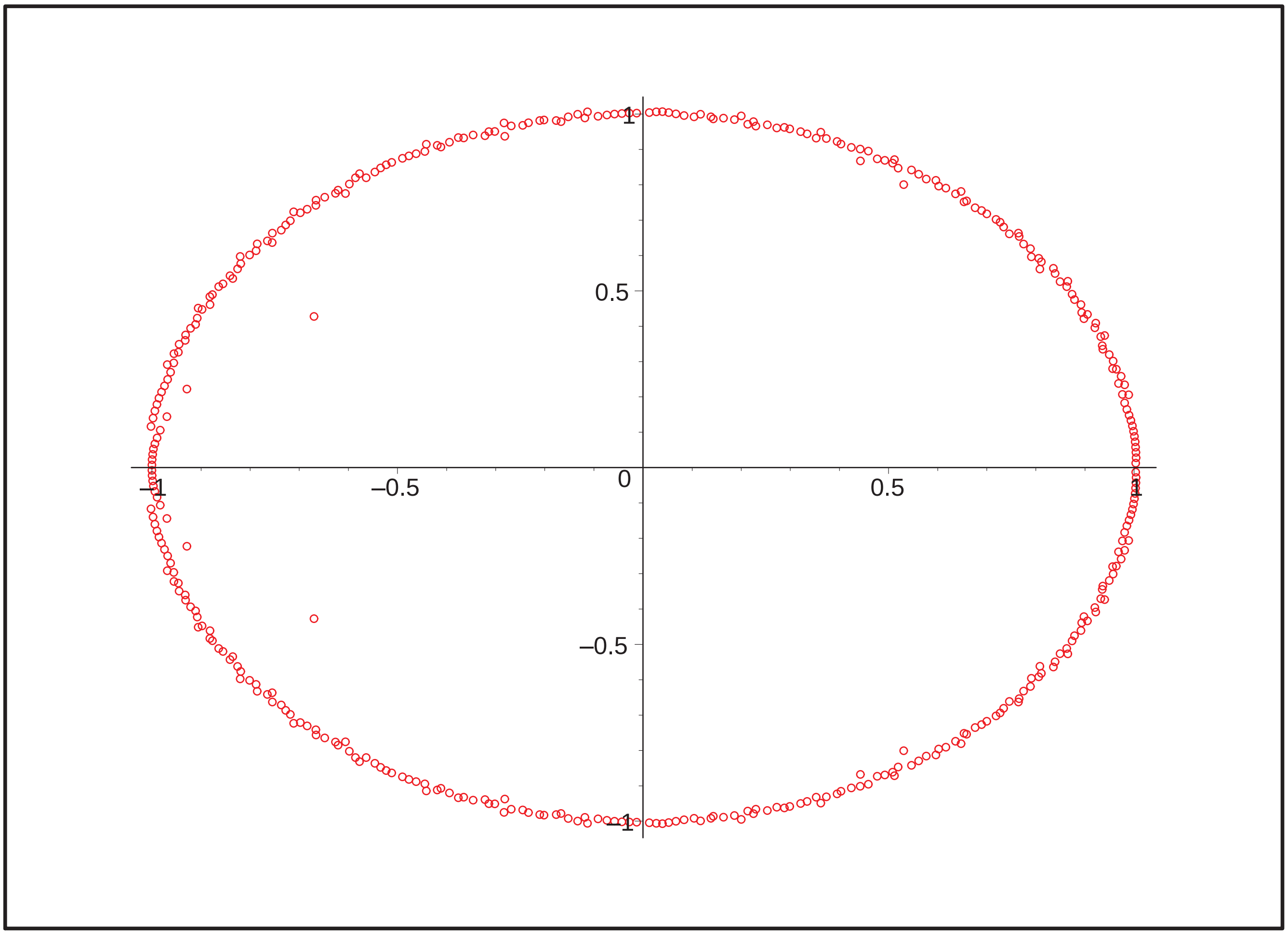
”You do not see from Fig. 4 in my first paper of 1799, how much work was required for a proper drawing of that curve.”

K. F. Gauss (1777-1855)

Fig. 4.



Gauss Werke Band III. Seite 30.



All zeros of $1/2 + x + x^4 + \ldots + x^{400}$

These packages deal most successfully with algebraic problems while many (perhaps most) serious applications require analytic objects such as definite integrals, series and differential equations.

All the elementary notions of analysis, like continuity and differentiability have to be given precise computational meaning.

Many of the difficult problems involve automatic simplification of complicated analytic formulae and recognition of when two very different such expressions represent the same object.

The packages can now substantially deal with large parts of the standard mathematics curriculum (and can outperform most of our undergraduates).

There is a coherent argument that they are the most significant part in a paradigm shift in how mathematics is done and certainly they have become a central research tool in many subareas of mathematics both from an exploratory and from a formal point of view.

$$\int (log(x)^7)dx =$$

$$(\ln(x))^7 x - 7x(\ln(x))^6 + 42x(\ln(x))^5$$

$$-210x(\ln(x))^4 + 840x(\ln(x))^3$$

$$-2520x(\ln(x))^2 + 5040x\ln(x) - 5040x$$

$$\int (exp(x) * log(x))dx =$$

$$e^x \ln(x) + \mathrm{Ei}(1, -x)$$

$$\int (exp(x)*log(x)^2)dx = \int (exp(x)*log(x)^2)dx$$

(It is acceptable now to see a line in a proof that says something like ”by a large calculation in Maple we see”.)

The first objective of symbolic algebra packages was to do as much exact mathematics as possible.

$$x^{198} - x^{197} - x^{181} + x^{174} + x^{168} - x^{157} - x^{133} + x^{115} + x^{112} - x^{86} - x^{83} + x^{65} + x^{41} - x^{30} - x^{24} + x^{17} + x - 1$$

Maple will factor this in a few seconds. (This polynomial has $(1-x)^9$ as a factor.)

$$\sum_{i=1}^{\infty} i^{-6} = \frac{1}{945}\,\pi^6$$

$$\sum_{i=1}^{\infty} \frac{1}{i^6\,(i+1)^6} =$$

$$-462 + \frac{2}{945}\,\pi^6 + \frac{7}{15}\,\pi^4 + 42\,\pi^2$$

$$\sum_{i=1}^{\infty} \frac{1}{i^3+1} = -1/3 + 1/3\,\gamma +$$

$$1/3\,\left(1/2 - 1/2\,\sqrt{-3}\right)\Psi(1/2+1/2\,\sqrt{-3})$$

$$+1/3\,\left(1/2 + 1/2\,\sqrt{-3}\right)\Psi(1/2-1/2\,\sqrt{-3})$$

A second objective is do it fast and to deal (in an arbitrary precision environment) with the more usual algorithms of analysis.

Basically one would like to be able to incorporate the usual methods of numerical analysis into an exact environment or at least into an infinite precision environment.

The problems are obvious and hard.

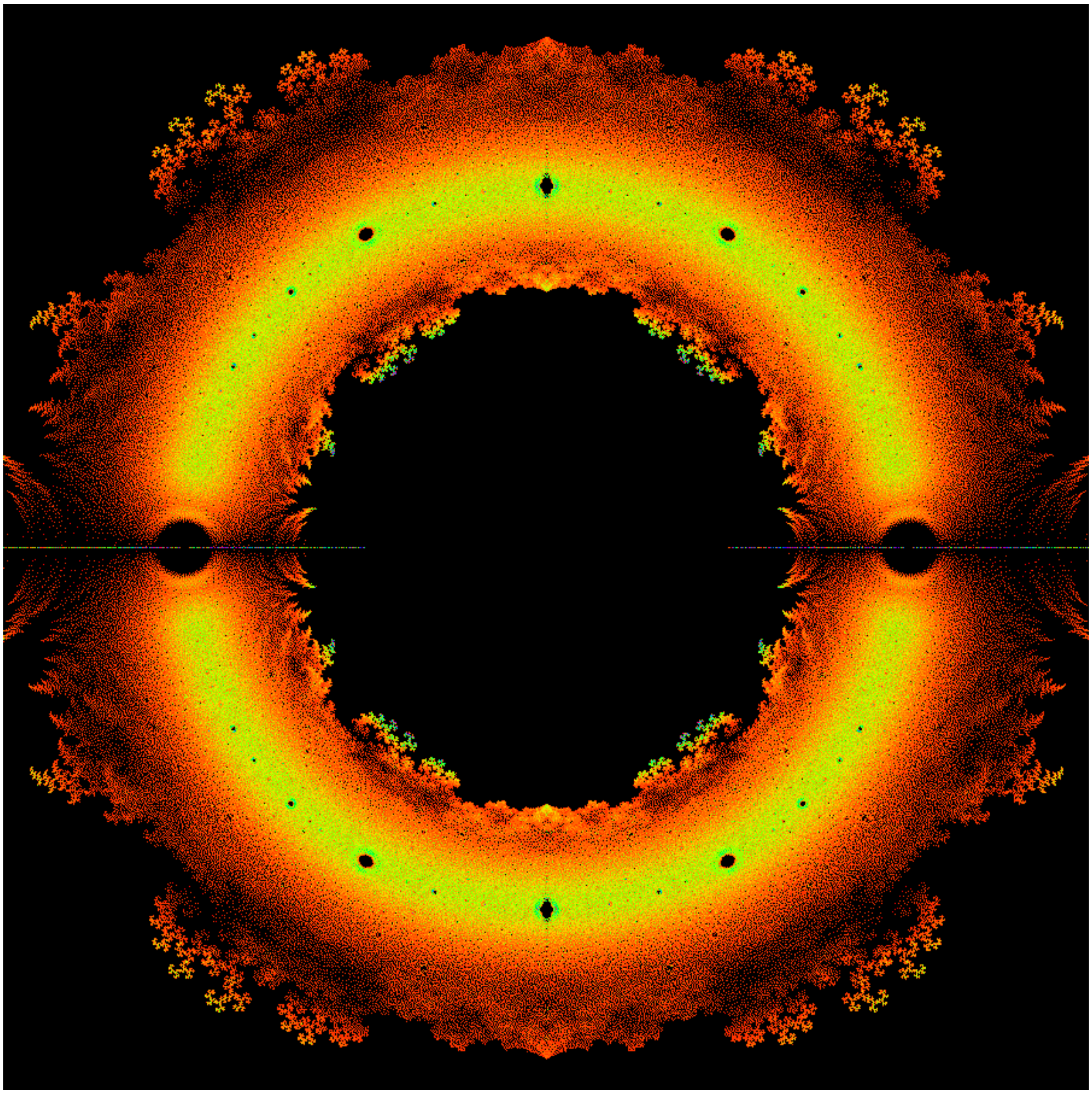
For example how does one do arbitrary precision numerical quadrature?

When does one switch methods with precision required or with different analytic properties of the integrand?

How does one deal with branch cuts of analytic functions? How does one even deal consistently with log (even this isn't completely worked out)?

More ambitiously how does one do a similar analysis for differential equations?

The goal is to marry the algorithms of analysis with symbolic and exact computation and to do this with as little loss of speed as possible. (SNAP)



Within this context a number of very interesting problems concerning the visualization of mathematics arise (how does one actually ”see” what one is doing).

It has been argued that Cartesian graphing was the most important invention of the last millennium.

Certainly it changed how we though about mathematics – the subsequent development of differential calculus rested on it.

More subtle and complicated graphics, like those of fractals, allow for a kind of exploration that was previously impossible.

There are many issues to be worked out here that live at the interface of mathematics, pedagogy and even psychology but are very timely to get right. (Think of how one visualizes the human genome and its patterns – which is after all just a particular several billion digit number base four.)

π RevEng

The Reverse Engineering Calculator

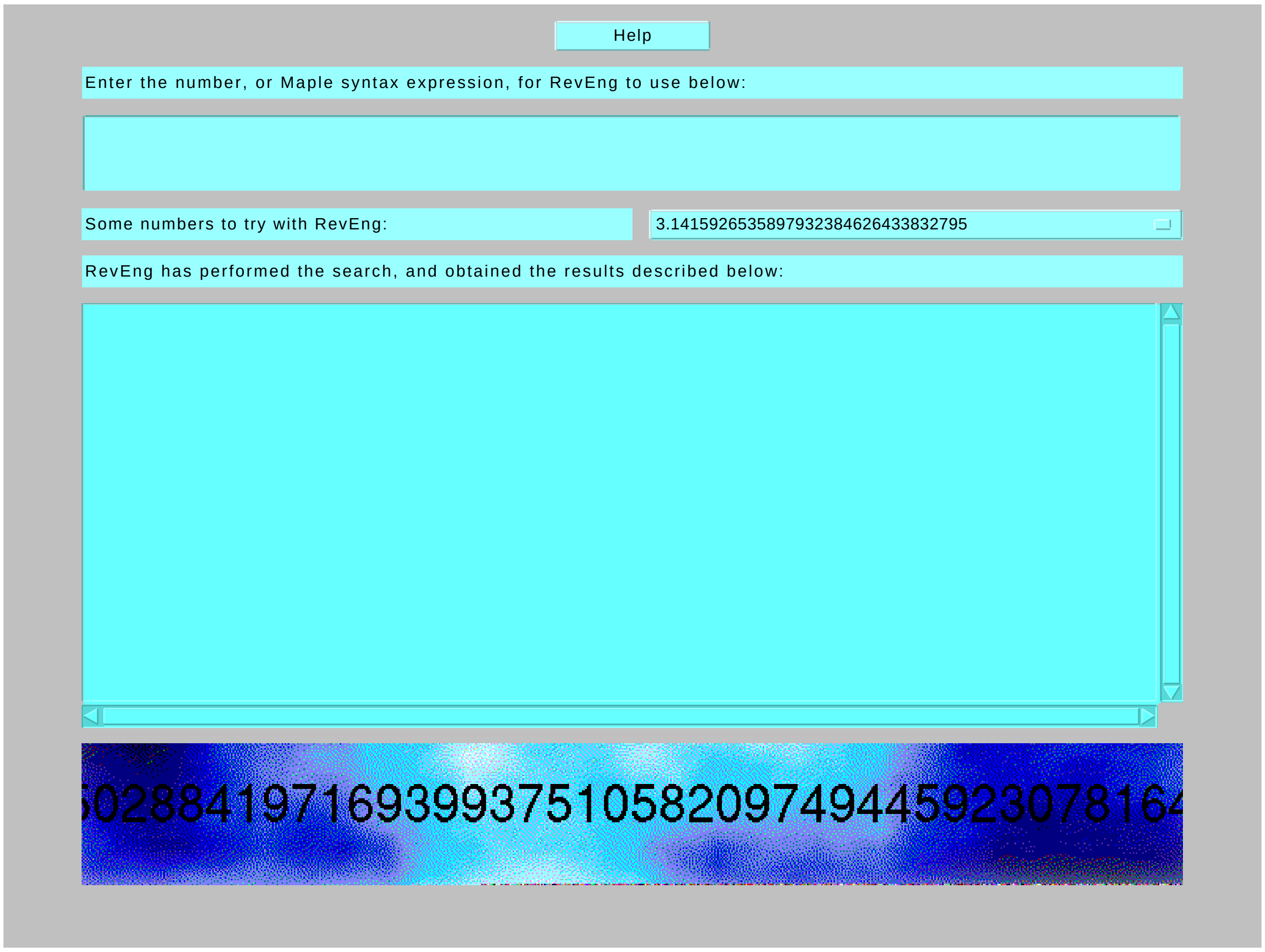
mitacs

Waterloo Maple

Centre for Experimental and Constructive Mathematics, Simon Fraser University

ADVANCING MATHEMATICS

This is a "Reverse Engineering Calculator" using Maple as a symbolic engine. Numeric-Symbolic searches are done with a hierarchy of algorithms, beginning with rational numbers. Results are given in exact Maple format when found.



Enter the number, or Maple syntax expression, for RevEng to use below:

Some numbers to try with RevEng: 3.1415926535897932384626433832795

RevEng has performed the search, and obtained the results described below:

The nature of our project is to attack those particular pieces of the puzzle that we see as important; where we have the appropriate skills; and which hold mathematical attraction.

The speed with which new research can be incorporated is striking and in this sense the technology transfer is close to immediate.

Scientific breakthroughs have various flavors.

A thousand fold speed up in large sparse large matrix multiplication is one type of breakthrough (as Maple has achieved between its last two versions).

Brand new algorithms are another obvious type of breakthrough (LLL).

The incremental building of large tools that in the end solve a large variety of problems.

This is not very sexy but is very central and sometimes very effective (redolent of the successful treating of childhood leukemia.)

Specific places where we have worked

– polynomial GCDs and resultant

– sparse simplex type methods

– simplification of elementary functions (unwinding number problems)

– infinite precision quadrature

– exact solutions of ODE’s

– an algebraic solver for DE’s

– fast arbitrary precision multiplication

– integer relation algorithms and inverse symbolic computation (PSLQ)

Projects we would like to pursue:

– zeros of arbitrary analytic functions in arbitrary regions

– visualization tools

– specialized mathematical/engineering tool boxes