SOME DISTINCTIVE INEQUALITIES

Peter Borwein and Tamás Erdélyi
Simon Fraser University

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Polynomials and Rational Functions.

The polynomials of degree at most n with complex coefficients:

$$\mathcal{P}_n^c := \left\{ p : p(z) = \sum_{i=0}^{n} a_i z^i, \quad a_i \in \mathbb{C} \right\}.$$

With real coefficients:

$$\mathcal{P}_n := \left\{ p : p(z) = \sum_{i=0}^{n} a_i z^i, \quad a_i \in \mathbb{R} \right\}$$

Rational functions of type (n, m) with complex coefficients:

$$\mathcal{R}_{m,n}^c := \left\{ p_m/q_n : p_m \in \mathcal{P}_m^c, q_n \in \mathcal{P}_n^c \right\}$$

With real coefficients:

$$\mathcal{R}_{m,n} := \left\{ p_m/q_n : p_m \in \mathcal{P}_m, q_n \in \mathcal{P}_n \right\}.$$

Classical Polynomial Inequalities.

Remez Inequality. *The inequality*

$$\|p\|_{[-1,1]} \le T_n\left((2+s)/(2-s)\right)$$

holds for every $p \in \mathcal{P}_n$ *and* s *satisfying*

$$m\left(\{x \in [-1,1] : |p(x)| \le 1\}\right) \ge 2-s.$$

Here T_n *is the Chebyshev polynomial:*

$$T_n(x) := \cos(n \arccos x).$$

Bernstein's Inequality. *For* $p \in \mathcal{P}_n^c$

$$|p'(x)| \le \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]}, \qquad -1 < x < 1.$$

Markov's Inequality. *For* $p \in \mathcal{P}_n^c$

$$\|p'\|_{[-1,1]} \le n^2 \|p\|_{[-1,1]}.$$

• The L_p analogue of Markov's Inequality states that

$$\int_{-1}^{1} |Q'(x)|^p dx \leq c^{p+1} n^{2p} \int_{-1}^{1} |Q(x)|^p dx$$

for every $Q \in \mathcal{P}_n$ and $0 < p < \infty$, where c is an absolute constant.

• We will prove this more generally with a constant 12.

• The best possible Markov factor in L_p is still an open problem even for $p = 2$ or $p = 1$.

Müntz Systems (Dirichlet Sums).

The system

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \quad \text{on } [0,1]$$

is called a *Müntz systems.*

Orthonormal Müntz-Legendre Polys.

We can explicitly orthonormalize

$$\mathrm{span}\{x^{\lambda_0}, \ldots , x^{\lambda_n}\}$$

Define L_n^*, the n-th orthonormal Müntz-Legendre polynomial by

$$L_n(x) := \frac{1}{2\pi i} \int_\Gamma \quad \prod_{k=0}^{n-1} \frac{t + \overline{\lambda}_k + 1}{t - \lambda_k} \frac{x^t dt}{t - \lambda_n}$$

$$= \sum_{k=0}^{n} c_{k,n} x^{\lambda_k}, \qquad x \in (0, \infty)$$

with

$$c_{k,n} := \frac{\prod_{j=0}^{n-1} (\lambda_k + \overline{\lambda}_j + 1)}{\prod_{j=0, j\neq k}^{n} (\lambda_k - \lambda_j)}$$

and

$$L_n^* := (1 + \lambda_n + \overline{\lambda}_n)^{1/2} L_n.$$

Then we get an orthonormal system. So

$$\int_0^1 L_n^*(x) \overline{L_m^*(x)} dx = \delta_{m,n}, \quad m, n = 0, 1, \ldots .$$

• One can base a very simple proof of Müntz's Theorem on this.

Müntz's Theorem. *For* $\lambda_i \geq 1$

$$\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

is dense in $C[0,1]$ *in the uniform norm if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

Conjecture (Full Müntz in L_p). *Let* $p \in [1, \infty]$. *Suppose* $\{\lambda_i\}_{i=0}^{\infty}$ *is a sequence of distinct real numbers greater than* $-1/p$. *Then*

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

is dense in $L_p[0,1]$ *if and only if*

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \infty.$$

Full Müntz in $C[0,1]$. (B&E). *Suppose* $\{\lambda_i\}_{i=1}^\infty$ *is a sequence of distinct, positive real numbers. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *if and only if*

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2+1} = \infty.$$

- The L_1 and L_2 cases also hold.

Müntz-Chebyshev polynomials.

In principal it is possible to construct an analogue of the Chebyshev Polynomial for a Müntz System

$$\mathrm{span}\{x^{\lambda_0}, \dots, x^{\lambda_n}\}$$

This will be an equioscillating “polynomial” and will be extremal for a number of problems

One needs these in the proof of the Full Müntz Theorem.

In particular one needs the characterization of denseness of an infinite Markov system in terms of denseness of the zeros of the associated Chebyshev polynomials.

Inequalities in Müntz Spaces.

We first present a simplified version of Newman's beautiful proof of a Markov-type inequality for Müntz polynomials. This modification allows us to prove the L_p analogues of Newman's Inequality.

Newman's Inequality. *Let* $\{\lambda_i\}_{i=0}^{\infty}$ *be a sequence of distinct nonnegative real numbers. Then*

$$\frac{\|xp'(x)\|_{[0,1]}}{\|p\|_{[0,1]}} \leq 9 \sum_{j=0}^{n} \lambda_j$$

for every p *in the linear span of*

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

In L_p *we must replace the constant 9 by 13 .*

For $p \geq 1$ and $P \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ with exponents λ_j greater than $-1/p$.

Sharp Markov Inequality. (B&E)

$$\|xP'(x)\|_{L_p[0,1]} \leq$$

$$13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right) \|P\|_{L_p[0,1]}$$

Nikolskii-type Inequality. (B&E)

$$\|y^{1/p} P(y)\|_{L_\infty[0,1]} \leq$$

$$13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right)^{1/p} \|P\|_{L_p[0,1]}$$

• Note the implication for Müntz's Theorem with exponents tending to $-1/p$.

Proof of modified Newman. It is equivalent to prove that

$$\frac{\|p'\|_{[0,\infty)}}{\|p\|_{[0,\infty)}} \le 9 \sum_{j=0}^{n} \lambda_j$$

for every p in $E_n(\Lambda)$ the linear span of

$$\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}\}.$$

WLOG we assume that $\sum_{j=0}^{n} \lambda_j = 1$.

The key is to examine

$$U(t) := \frac{1}{2\pi i} \int_\Gamma \frac{e^{-zt}}{(1-z)B(z)} dz$$

where

$$B(z) := \prod_{j=1}^{n} \frac{z-\lambda_j}{z+\lambda_j}$$

and

$$\Gamma := \{z \in \mathbb{C} : |z-1| = 1\}.$$

Now if $P \in E_n(\Lambda)$ is of the form

$$P(t) = \sum_{j=0}^{n} c_j e^{-\lambda_j t}, \quad c_j \in \mathbb{R}.$$

Then

$$\int_0^\infty P(t+a)U''(t)dt$$

$$= 3P(a) - P'(a).$$

Therefore

$$|P'(a)| \le 3|P(a)| + \int_0^\infty |P(t+a)\,U''(t)|dt.$$

and

$$\|P'\|_{[0,\infty)} \le 3\,\|P\|_{[0,\infty)} + 6\,\|P\|_{[0,\infty)}. \quad \square$$

- The constant should be 4?

Variant of Newman. (B&E).

If $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence with $\lambda_0 = 0$ and

$$\lambda_{i+1} - \lambda_i \geq 1$$

then

$$\|p'\|_{[0,1]} \leq 18 \left(\sum_{j=1}^{n} \lambda_j \right) \|p\|_{[0,1]}$$

for every p in the linear span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}.$$

•The gap condition is needed to make this work.

• An irritating feature of these inequalities is that the proofs don't translate off the interval $[0,1]$. Though the inequalities do.

Newman on $[a,b], a>0$. (B&E). *Let $\{\lambda_i\}_{i=1}^\infty$ be a sequence of nonnegative real numbers. Assume that there exists a $\delta > 0$ so that*

$$\lambda_i \geq \delta i$$

for each i. Then there exists a constant $c(a,b,\delta)$ depending only on a, b, and δ so that

$$\|P'\|_{[a,b]} \leq c(a,b,\delta) \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[a,b]}$$

for P in the span of $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$.

Lorentz's Problem for Müntz Polys.

Conjecture.

$$\sup_p \frac{|\, p'(1) \,|}{\|p\|_{[0,2]}} \leq ??$$

where the sup is over all Müntz polynomials

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j \geq 0$$

independent of the exponents.

• These following results improve inequalities of Lorentz and Schmidt and others going back 25 years.

• Lorentz now conjectures the above with a $C * n$ bound?

Theorem. *For every* $0 < a < b$

$$\sup_p \frac{|\, p'(a) \,|}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}$$

The sup is over all Müntz polynomials

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j \geq 0.$$

Theorem. *for* $\delta \in (0, (b-a)/2)$

$$\|p'\|_{[a+\delta, b-\delta]} \leq 4(n+2)^3 \, \delta^{-1} \|p\|_{[a,b]}$$

where

$$p(t) = a_0 + \sum_{i=1}^{n} a_j e^{\lambda_j t}, \;\; a_j, \lambda_j \in \mathbb{R}.$$

A Remez Inequality for Müntz Spaces.

• This Remez-type inequality allows us to resolve two reasonably long standing conjectures.

• The first, due to D. J. Newman and dating from 1978, asserts that if

$$\sum_{i=1}^{\infty} 1/\lambda_i < \infty$$

then the set of products

$$\{p_1 p_2 : p_1, p_2 \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}\}$$

is not dense in $C[0,1]$.

• The second is a complete extension of Müntz's classical theorem on the denseness of Müntz spaces in $C[0,1]$ to denseness in $C[A]$, where $A \subset [0,\infty)$ is an arbitrary compact set with positive Lebesgue measure.

Müntz's Theorem Generalized. *For an arbitrary compact set $A \subset [0, \infty)$ with positive Lebesgue measure,*

$$span\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \qquad \lambda_i \geq 1$$

is dense in $C[A]$ if and only if

$$\sum_{i=1}^{\infty} 1/\lambda_i = \infty.$$

- Let

$$p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ The most useful form of our Remez inequality states:

Bounded Remez Inequality. (B&E).

For every sequence $\{\lambda_i\}_{i=0}^{\infty}$ *satisfying*

$$\sum_{i=1}^{\infty} 1/\lambda_i < \infty$$

there is a constant c *depending only on* $\{\lambda_i\}_{i=0}^{\infty}$ *and* s *(and not on* n*,* ϱ*, or* A*) so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every Müntz polynomial p*, as above, associated with* $\{\lambda_i\}_{i=0}^{\infty}$*, and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* $s > 0$*.*

Arc Length of the Lemniscate $\{|p(z)| = 1\}$.

•In 1958 Erdös, Herzog and Piranian conjectured that the lemniscate

$$E_n := E_n(p) := \{z \in \mathbb{C} : |p(z)| = 1\}$$

where p is a monic polynomial of degree n, so

$$p(z) := \Pi_{i=1}^{n}(z - \alpha_i) \qquad \alpha_i \in \mathbb{C}.$$

is of maximum length for $p(z) := z^n - 1$. (Which is of length $2n + 0(1)$.)

• Best partial to date, due to Pommerenke, shows that the maximum length is at most $74n^2$.

• It carries a cash prize from Erdös of \$250.

Theorem (P.B.). *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Then*

$$E_n := \{z \in \mathbb{C} : |\Pi_{i=1}^{n}(z - \alpha_i)| = 1\}$$

has length at most $8\pi e n \, (\leq 69n)$.

• This at least gives the right order of growth.

• The proof relies on two classical theorems. One by Cartan and one by Poincaré.

Cartan's Lemma. *If*

$$p(z) := \prod_{i=1}^{n} (z - \alpha_i)$$

then the inequality

$$|p(z)| > 1$$

holds outside at most n circular discs, the sum of whose radii is at most $2e$.

Poincaré's Formula. *Let* Γ *be a rectifiable curve contained in* $\mathbb{S}$ *(the Riemann sphere). Let* $v(\Gamma, x)$ *denote the number of times that a great circle consisting of points equidistant from the antipodes* $\pm x$ *intersects* Γ*. Then the length of* Γ*,* $L_{\mathbb{S}}(\Gamma)$*, is given by*

$$L_{\mathbb{S}}(\Gamma) = \frac{1}{4} \int_{\mathbb{S}} v(\Gamma, x) dx$$

where dx *is area measure on* $\mathbb{S}$*.*

Corollary. *Suppose* Γ *is an algebraic curve in* $\mathbb{R}^2$ *of degree at most* N *and* D *is a disc of radius* R*. Then the length of* $\Gamma \cap D$ *is at most* $2\pi RN$*.*

Sharp Extensions of Bernstein's Inequality to Rational Spaces.

Let

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; A)$$

$$:= \left\{ \frac{p_n(z)}{\prod_{j=1}^{n}(z-a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

where the A indicates that the poles are to avoid A.

• If the a_i tend to infinity we recover the ordinary polynomials. So the following results are sharp extensions of the usual Bernstein inequality.

•These are also sharp extensions of results of Russak and others.

Bernstein-Szegő Type Inequality. (B&E).

For $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$, *let*

$$B_n(x) := \sum_{k=1}^n \mathrm{Re}\left(\frac{\sqrt{a_k^2-1}}{a_k - x}\right)$$

where the root $\sqrt{a_k^2-1}$ *is determined by*

$$c_k := a_k - \sqrt{a_k^2-1}, \qquad |c_k| < 1.$$

Then

$$(1-x^2)f'(x)^2 + B_n(x)^2 f(x)^2 \le B_n(x)^2 \|f\|_{[-1,1]}^2$$

and

$$|f'(x)| \le \frac{1}{\sqrt{1-x^2}} B_n(x) \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$.

Theorem. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$. *Then*

$$|f'(z_0)|/\|f\|_{\partial D} \leq$$

$$\max \left\{ \sum_{\substack{j=1 \\ |a_j|>1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j|<1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$.

Theorem. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(x_0)|/\|f\|_{\mathbb{R}} \leq$$

$$\max \left\{ \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)>0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2}, \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)<0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2} \right\}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \mathbb{R})$.

Inequalities for p_n'/p_n.

These are metric inequalities of the form

$$m\left(\left\{x : \frac{r_n'(x)}{r_n(x)} \geq \alpha\right\}\right) \leq \frac{\beta \cdot n}{\alpha}, \qquad \alpha > 0$$

where r_n is a rational function of type (n, n) and β is a constant independent of n. Here m is Lebesgue measure.

Theorem (Loomis). *If $p_n \in \mathcal{P}_n$ has n real roots then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p_n'(x)}{p_n(x)} \geq \alpha\right\}\right) = \frac{n}{\alpha} \quad \textit{for } \alpha > 0$$

Proof. Draw a picture of

$$\frac{p_n'(x)}{p_n(x)}$$

Lemma. *If $p_n \in \mathcal{P}_n$ is positive on $[a,b]$ then there exists q_n, $s_n \in \mathcal{P}_n$ nonnegative on $[a,b]$ with all real roots (in $[a,b]$) so that $p_n(x) = q_n(x) + s_n(x)$.*

Theorem. *Let $p_n \in \mathcal{P}_n$ then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p_n'(x)}{p_n(x)} \geq \alpha\right\}\right) \leq \frac{2n}{\alpha}, \qquad \alpha > 0.$$

Theorem. *If $r_n = p_n/q_n \in \mathcal{R}_{n,n}n$ then*

$$m\left(\left\{x \in \mathbb{R} : \frac{r_n'(x)}{r_n(x)} \geq \alpha\right\}\right) \leq \frac{8n}{\alpha}, \qquad \alpha > 0.$$

• It would be interesting to know the right constant above. It might well be 2π This is closely related to the incomplete rational problem concerning the interval of denseness of

$$\{\exp(-nx)p_n(x)/q_n(x)\}.$$