A SHARP BERNSTEIN-TYPE INEQUALITY FOR EXPONENTIAL SUMS

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We prove the "right" Bernstein-type inequality for exponential sums

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{R} \right\}$$

Schmidt proved that there is a constant $c(n)$ so that

$$\|f'\|_{[a+\delta, b-\delta]} \le c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$.

Lorentz replaced $c(n)$ by $c(\alpha)n^{\alpha \log n}$ (Xu improved this to allow $\alpha = \frac{1}{2}$).

Lorentz speculated that $c(n)$ should be cn.

We proved a weaker version of this with cn^3 instead of cn.

Our main result is

$$\sup_{0\neq f\in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a,b-y\}}.$$

In this Bernstein-type inequality even the pointwise factor is sharp up to a multiplicative absolute constant; so

$$\frac{1}{e-1}\frac{n-1}{\min\{y-a,b-y\}} \leq \sup_{0\neq f\in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}$$

The critical inequality is

$$\sup_{0\neq f\in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n-1$$

where

$$\widetilde{E}_{2n} := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} \left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t} \right) \right\}.$$

Denote by $\mathcal{P}_n$ the set of all polynomials of degree at most n with real coefficients.

Proposition (Bernstein's Inequality). *The inequality*

$$|p'(x)| \le \frac{n}{\sqrt{1-x^2}}\,\|p\|_{[-1,1]}, \qquad -1 < x < 1$$

holds for every $p \in \mathcal{P}_n$.

This implies by simple substitution and scaling

$$|f'(y)| \le \frac{2n}{\min\{y-a, b-y\}}\,\|f\|_{[a,b]}, \qquad y \in (a,b)$$

holds for the particular exponential sums

$$f(t) = a_0 + \sum_{j=1}^{n} a_j e^{jt}, \quad a_j \in \mathbb{R}.$$

This is a very special case ($\lambda_j = j$) of our Theorem.

The following slight improvement of Bernstein's inequality may be found in Natanson.

Proposition. *The inequality*

$$|p'(0)| \le (2n-1)\,\|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_{2n}$.

This gives

Proposition. *The inequality*

$$|p'(x)| \le \frac{2n-1}{1-|x|}\,\|p\|_{[-1,1]}, \qquad x \in (-1,1)$$

holds for every $p \in \mathcal{P}_{2n}$.

2. Notation and Definitions

The notations

$$\|f\|_A := \sup_{x \in A} |f(x)|$$

and

$$\|f\|_{L_p(A)} := \left(\int_A |f|^p \right)^{1/p}$$

are used throughout. Also

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

$$\widetilde{E}_{2n} := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} \left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t} \right), \quad a_j, b_j, \lambda_j \in \mathbb{R} \right\}$$

$$E_n^* := \left\{ f : f(t) = \sum_{j=1}^{l} P_{k_j}(t) e^{\lambda_j t}, \quad \sum_{j=1}^{l} (k_j + 1) = n \right\}$$

where $\mathcal{P}_k$ denotes the set of all polynomials of degree at most k with real coefficients.

3. New Results

Theorem A. *We have*

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n - 1.$$

Theorem B. *The inequality*

$$\sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}.$$

holds for every $n \in \mathbb{N}$ and $y \in (a,b)$.

Theorem C. *The inequality*

$$\frac{1}{e-1} \frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}$$

holds for every $n \in \mathbb{N}$ and $y \in (a,b)$.

Theorem D. *The inequality*

$$\|f\|_{[a+\delta,b-\delta]} \le 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

holds for all $f \in E_n$, $p \in (0,2]$, *and* $\delta \in \big(0, \frac{1}{2}(b-a)\big)$.

4. Chebyshev and Descartes Systems

Definition. *The system* $(f_0, \dots, f_n)$ *is said to be a Descartes system (or order complete Chebyshev system) on an interval* I *if each* $f_i \in C(I)$ *and*

$$D\begin{pmatrix} f_{i_0} & f_{i_1} & \dots & f_{i_m} \\ x_0 & x_1 & \dots & x_m \end{pmatrix} > 0$$

for any $0 \le i_0 < i_1 < \cdots < i_m \le n$ *and* $x_0 < x_1 < \cdots < x_m$ *from* I.

This is a property of the basis. It implies that any finite dimensional subspace generated by some basis elements is a Chebyshev space on I.

Lemma. *The system*

$$(e^{\lambda_0 t}, e^{\lambda_1 t}, \dots), \qquad \lambda_0 < \lambda_1 < \cdots$$

is a Descartes system on $(-\infty, \infty)$. *In particular, it is also a Chebyshev system on* $(-\infty, \infty)$.

Proof. See, for example, Karlin and Studden. □

The following lemma is crucial

Lemma. *Suppose* $0 < \lambda_0 < \lambda_1 < \cdots$. *Then*

$$(\sinh \lambda_0 t, \sinh \lambda_1 t, \dots)$$

is a Descartes system on $(0, \infty)$.

5. The Pinkus-Smith Comparison Theorem

Proposition. *Suppose* $(f_0, \dots, f_n)$ *is a Descartes system on* $[a, b]$*. Suppose*

$$p = f_\alpha + \sum_{i=1}^{k} a_i f_{\lambda_i}, \qquad q = f_\alpha + \sum_{i=1}^{k} b_i f_{\gamma_i}$$

where $0 \le \lambda_1 < \cdots < \lambda_k$, $0 \le \gamma_1 < \cdots < \gamma_k$,

$$0 \le \gamma_i \le \lambda_i < \alpha, \quad i = 1, 2, \dots, m$$

and

$$\alpha < \lambda_i \le \gamma_i, \quad i = m+1, m+2, \dots, k$$

with strict inequality for at least one index . If

$$p(x_i) = q(x_i) = 0, \qquad i = 1, 2, \dots, k$$

where $x_i \in [a, b]$ *are distinct, then*

$$|p(x)| \le |q(x)|$$

for all $x \in [a, b]$ *with strict inequality for* $x \ne x_i$.

6. Chebyshev Polynomials

Suppose

$$H_n := \text{span}\{f_0, f_1, \dots , f_n\}$$

is a Chebyshev space on $[a, b]$ and A is a compact subset of $[a, b]$.

We define the *generalized Chebyshev polynomial*

$$T_n := T_n\{f_0, f_1, \dots , f_n; A\}$$

for H_n on A by the following three properties:

$$T_n \in \text{span}\{f_0, f_1, \dots , f_n\}$$

there is an alternation set $(x_1 < x_2 < \cdots < x_n)$

$$|T_n(x_i)| = \|T_n\|_A, \qquad i = 0, 1, \dots , n$$

with $\quad \text{sign}(T_n(x_{i+1})) = -\text{sign}(T_n(x_i)), \qquad i = 0, 1, \dots , n-1$

and

$$\|T_n\|_A = 1 \quad \text{with} \quad T_n(\max A) > 0.$$

The Chebyshev polynomials T_n for H_n on A encode much of the information of how the space H_n behaves with respect to the uniform norm on A. Many extremal problems are solved by the Chebyshev polynomials.

When $(f_0, f_1, \dots)$ is a Markov system on $[a,b]$ we can introduce the sequence $(T_n)_{n=0}^{\infty}$ of *associated Chebyshev polynomials*

$$T_n := T_n\{f_0, f_1, \dots, f_n; [a,b]\}$$

for H_n on $[a,b]$. Then $(T_0, T_1, \dots)$ is a Markov system on $[a,b]$ again with the same span.

Proposition. *Suppose $H_n := \operatorname{span}\{f_0, \dots, f_n\}$ is a Chebyshev space on $[a,b]$ with associated Chebyshev polynomial*

$$T_n := T_n\{f_0, f_1, \dots, f_n; [a,b]\}$$

and each f_i is differentiable at b. Then

$$\max\{|p'(b)| : p \in H_n, \ \ \|p\|_{[a,b]} \le 1, \ \ p(b) = T_n(b)\}$$

is attained by T_n.

Proposition (Lexicographic Property). *Let $(f_0, f_1, \dots)$ be a Descartes system on $[a,b]$. Suppose $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ and $\gamma_0 < \gamma_1 < \cdots < \gamma_n$ are nonnegative integers satisfying*

$$\lambda_i \le \gamma_i, \qquad i = 0, 1, \dots, n.$$

Let

$$T_n := T_n\{f_{\lambda_0}, f_{\lambda_1}, \dots, f_{\lambda_n}; [a,b]\}$$

and

$$S_n := T_n\{f_{\gamma_0}, f_{\gamma_0}, \dots, f_{\gamma_n}; [a,b]\}$$

denote the associated Chebyshev polynomials. Let

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n \quad \text{and} \quad \beta_1 < \beta_2 < \cdots < \beta_n$$

denote the zeros of T_n and S_n, respectively. Then

$$\alpha_i \le \beta_i, \quad i = 1, 2, \dots, n$$

with strict inequality if $\lambda_i \ne \gamma_i$ for at least one index i. (In other words, the zeros of T_n lie to the left of the zeros of S_n.)

7. A Comparison Theorem

The heart of the proof of Theorem is part e] of the following comparison theorem, which can be proved by a very subtle zero counting argument.

Theorem. *Let*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n, \quad 0 < \gamma_0 < \gamma_1 < \cdots < \gamma_n.$$

Suppose $\lambda_i \le \gamma_i$ *for each* i. *Let*

$$H_n := \mathrm{span}\{\sinh \lambda_0 t, \sinh \lambda_1 t, \ldots, \sinh \lambda_n t\}$$

and

$$G_n := \mathrm{span}\{\sinh \gamma_0 t, \sinh \gamma_1 t, \ldots, \sinh \gamma_n t\}.$$

Denote the associated Chebyshev polynomials for H_n *and* G_n *on* $[0,1]$ *by*

$$T_{n,\lambda} := T_n\{\sinh \lambda_0 t, \sinh \lambda_1 t, \ldots, \sinh \lambda_n t; [0,1]\}$$

and

$$T_{n,\gamma} := T_n\{\sinh \gamma_0 t, \sinh \gamma_1 t, \ldots, \sinh \gamma_n t; [0,1]\}$$

respectively. The following statements hold.

a] *Let*

$$\alpha_1 < \alpha_2 < \ldots < \alpha_n \quad \textit{and} \quad \beta_1 < \beta_2 < \cdots < \beta_n$$

denote the zeros of $T_{n,\lambda}$ *and* $T_{n,\gamma}$*, respectively. Then*

$$\alpha_i \leq \beta_i, \quad i = 1, 2, \ldots, n$$

(in other words, the zeros of $T_{n,\lambda}$ *lie to the left of the zeros of* $T_{n,\gamma}$*).*

b] *The value*

$$\max\{|p'(0)| : p \in H_n, \ \ \|p\|_{[0,1]} \leq 1\}$$

is attained uniquely by $\pm T_{n,\lambda}$*.*

c] *We have*

$$T_{n,\lambda}(1) = T_{n,\gamma}(1) = 1.$$

d] *We have*

$$|T'_{n,\lambda}(0)| \geq |T'_{n,\gamma}(0)|.$$

e] *We have*

$$\max_{0 \neq p \in H_n} \frac{|p'(0)|}{\|p\|_{[0,1]}} \geq \max_{0 \neq q \in G_n} \frac{|q'(0)|}{\|q\|_{[0,1]}}.$$

8 Proof of Theorem A

Theorem A. *We have*

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n - 1.$$

Proof of Theorem A. First we prove that

$$|f'(0)| \leq (2n-1)\,\|f\|_{[-1,1]}$$

for every $f \in \widetilde{E}_{2n}$. So let

$$f \in \operatorname{span}\{1, e^{\pm\lambda_1 t}, e^{\pm\lambda_2 t}, \dots, e^{\pm\lambda_n t}\}$$

with some non-zero real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, where, without loss of generality, we may assume that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

Let

$$g(t) := \tfrac{1}{2}(f(t) - f(-t)).$$

Observe that

$$g \in \operatorname{span}\{\sinh \lambda_1 t, \sinh \lambda_2 t, \dots, \sinh \lambda_n t\}.$$

It is also straightforward that

$$g'(0) = f'(0) \qquad \text{and} \qquad \|g\|_{[0,1]} \le \|f\|_{-1,1]}.$$

For a given $\epsilon > 0$, let

$$H_{n,\epsilon} := \operatorname{span}\{\sinh \epsilon t, \sinh 2\epsilon t, \dots, \sinh n\epsilon t\}$$

and

$$K_{n,\epsilon} := \sup\left\{|h'(0)| : h \in H_{n,\epsilon},\ \|h\|_{[0,1]} = 1\right\}.$$

By the comparison theorem it is sufficient to prove that $\inf\{K_{n,\epsilon} : \epsilon > 0\} \le 2n - 1$. Observe that every $h \in H_{n,\epsilon}$ is of the form

$$h(t) = e^{-n\epsilon t} P(e^{\epsilon t}), \qquad P \in \mathcal{P}_{2n}.$$

Therefore, using Bernstein, we obtain for every $h \in H_{n,\epsilon}$ that

$$\begin{aligned} |h'(0)| &= |\epsilon P'(1) - n\epsilon P(1)| \\ &\leq \frac{\epsilon(2n-1)}{1-e^{-\epsilon}} \|P\|_{[e^{-\epsilon},e^{\epsilon}]} + n\epsilon \|P\|_{[e^{-\epsilon},e^{\epsilon}]} \\ &\leq \left(\frac{\epsilon(2n-1)}{1-e^{-\epsilon}} + n\epsilon \right) e^{n\epsilon} \|h\|_{[-1,1]}. \end{aligned}$$

It follows that

$$K_{n,\epsilon} \leq \left(\frac{\epsilon(2n-1)}{1-e^{-\epsilon}} + n\epsilon \right) e^{n\epsilon}.$$

So $\inf\{K_{n,\epsilon} : \epsilon > 0\} \leq 2n-1$ as required.

Now we prove that

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} \geq 2n-1.$$

Let $\epsilon > 0$ be fixed. We define

$$Q_{2n,\epsilon}(t) := e^{-n\epsilon t} T_{2n-1} \left(\frac{e^{\epsilon t}}{e^{\epsilon}-1} - \frac{1}{e^{\epsilon}-1} \right)$$

where T_{2n} denotes the Chebyshev polynomial of degree $2n$ defined by

$$T_{2n-1}(x) = \cos(2n \arccos x), \quad x \in [-1, 1].$$

It is simple to check that $Q_{2n,\epsilon} \in \widetilde{E}_{2n}$,

$$\|Q_{2n,\epsilon}\|_{[-1,1]} \leq e^{n\epsilon t}$$

and

$$|Q'_{2n,\epsilon}(0)| \geq 2n - 1 - n\epsilon.$$

The result follows by letting $\epsilon > 0$ tend to 0. $\square$

The real Müntz's Theorem.

Müntz's Theorem. *For* $\lambda_i \geq 1$

$$\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *in the uniform norm if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

Full Müntz in $C[0,1]$. (B&E). *Suppose* $(\lambda_i)_{i=1}^{\infty}$ *is a sequence of distinct, positive real numbers. Then*

$$\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *if and only if*

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty.$$

- The L_1, L_2 and L_∞ cases also hold.

More Inequalities in Müntz Spaces.

Newman's Inequality. *Let* $\{\lambda_i\}_{i=0}^{\infty}$ *be a sequence of distinct nonnegative real numbers. Then*

$$\frac{\|xp'(x)\|_{[0,1]}}{\|p\|_{[0,1]}} \le 9 \sum_{j=0}^{n} \lambda_j$$

for every p *in the linear span of*

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

• *This also holds in in* L_p *where we must replace the constant 9 by 13 .*

For $p \geq 1$ and $P \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ with exponents λ_j greater than $-1/p$.

Sharp Markov Inequality. (B&E)

$$\|xP'(x)\|_{L_p[0,1]} \leq$$

$$13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right) \|P\|_{L_p[0,1]}$$

Nikolskii-type Inequality. (B&E)

$$\|y^{1/p} P(y)\|_{L_\infty[0,1]} \leq$$

$$13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right)^{1/p} \|P\|_{L_p[0,1]}$$

- Note the implication for Müntz's Theorem with exponents tending to $-1/p$.

- The constant should be 4?

Newman on $[a,b], a > 0$. (B&E). *Let* $(\lambda_i)_{i=1}^{\infty}$ *be a sequence of nonnegative real numbers. Assume that there exists a* $\delta > 0$ *so that*

$$\lambda_i \geq \delta i$$

for each i. *Then there exists a constant* $c(a,b,\delta)$ *depending only on* a, b, *and* δ *so that*

$$\|P'\|_{[a,b]} \leq c(a,b,\delta) \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[a,b]}$$

for P *in the span of* $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$.

Müntz's Theorem Generalized. *For an arbitrary compact set $A \subset [0, \infty)$ with positive Lebesgue measure,*

$$\mathit{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \qquad \lambda_i \geq 1$$

is dense in $C[A]$ if and only if

$$\sum_{i=1}^{\infty} 1/\lambda_i = \infty.$$

- Let

$$p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ The most useful form of our Remez inequality states:

Bounded Remez Inequality. (B&E).

For every sequence $(\lambda_i)_{i=0}^\infty$ *satisfying*

$$\sum_{i=1}^{\infty} 1/\lambda_i < \infty$$

there is a constant c *depending only on* $\{\lambda_i\}_{i=0}^\infty$ *and* s *(and not on* n, ϱ, *or* A*) so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every Müntz polynomial p, *as above, associated with* $(\lambda_i)_{i=0}^\infty$, *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* $s > 0$.