SOME OLD PROBLEMS ON POLYNOMIALS WITH INTEGER COEFFICIENTS

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Typeset by $\mathcal{AMS}$-TEX

• Some old chestnuts. All at least 35 years old.

• All involve Chebyshev type problems for polynomials with integer coefficients.

• All very hard.

• All have a highly non-trivial computational component.

• All have accessible partials?

• All very interesting.

A. The Integer Chebyshev Problem of Hilbert and Fekete.

1. Problem. Find

$$C_N[\alpha, \beta] :=$$

$$\left(\min_{a_i \in z, a_N \neq 0} \|a_0 + a_1 x + \ldots + a_N x^N\|_{[\alpha,\beta]} \right)^{\frac{1}{N}} .$$

We will restrict to $\beta - \alpha \leq 4$.

2. One can show that

$$C[\alpha, \beta] := \lim_{N \to \infty} C_n[\alpha, \beta]$$

exists. This is the integer Chebyshev constant for the interval or the integer transfinite diameter.

3. We (T. Erdélyi and P.B.) show

$$\frac{1}{2.3768- \in} \le C[0,1] \le \frac{1}{2.360}.$$

Lemma. *Suppose*

$$q_m(x) = a_m x^m + \ldots + a_0, \qquad a_m \in Z$$

has all its roots in $(0,1)$. *(That is:* $q_m \in TR(0,1)$*). Then, provided* $(q_m, p_n) = 1$

$$\|p_n\|_{[0,1]}^{1/n} \ge \frac{1}{a_m^{1/m}}.$$

• It is conjectured (Chudnovskys, Montgomery) that the lemma gives the right bound in 3. (This is likley false.)

B. The Schur, Siegel, Smyth Trace Problem.

1. Conjecture. Suppose

$$p_n(z) = a_n z^n + \ldots + a_0, a_i \in Z$$

has all real, positive roots and is irreducible. Then

$$a_{n-1} \geq (2-\epsilon)n.$$

2. Partials. Except for finitely many (explicit) exceptions

$a_{n-1} \geq e^{1/2}n$ Schur (1918)

$a_{n-1} \geq (1.733..)n$ Siegel (1943)

$a_{n-1} \geq (1.771..)n$ Smyth (1983).

3. The Relationship to the Small Interval Problem.

Lemma. *If*

$$C[0,1/m] \leq 1/(m+\delta)$$

then, for totally positive polynomials

$$a_{n-1} \geq \delta n$$

(with finitely many explicit exceptions).

Corollary. $\delta > 1.744$

Proof. By example on $C[0,1/100]$.

C. Prouhet-Tarry-Escott Problem.

1. Conjecture. For any N there exists $p \in Z[x]$ (a polynomial with integer coefficients) so that

$$p(x) = (x-1)^N q(x) = \Sigma a_k x^k$$

and

$$S(p) := \Sigma |a_k| = 2N.$$

Almost equivalently (though not quite obviously) this polynomial must have coefficients $\{0, -1, +1\}$ and so

$$\|p\|_{L^2\{|z|=1\}} = \sqrt{2N}.$$

2. The Basis for the Conjecture.

$x^{\alpha_1}+\ldots+x^{\alpha_N}-x^{\beta_1}-\ldots-x^{\beta_N}=0((x-1)^N)$.

For $N=2,\ldots,10$ with

$$[\alpha_1,\ldots,\alpha_N] \quad \text{and} \quad [\beta_1,\ldots,\beta_N]$$

- $[0,3]=[1,2]$
- $[1,2,6]=[0,4,5]$
- $[0,4,7,11]=[1,2,9,10]$
- $[1,2,10,14,18]=[0,4,8,16,17]$
- $[0,4,9,17,22,26]=[1,2,12,14,24,25]$
- $[0,18,27,58,64,89,101]$

$=[1,13,38,44,75,84,102]$

- $[0, 4, 9, 23, 27, 41, 46, 50]$

$= [1, 2, 11, 20, 30, 39, 48, 49]$

- $[0, 24, 30, 83, 86, 133, 157, 181, 197]$

$= [1, 17, 41, 65, 112, 115, 168, 174, 198]$

- $[0, 3083, 3301, 11893, 23314, 24186, 35607,$

$44199, 44417, 47500] =$

$[12, 2865, 3519, 11869, 23738, 23762, 35631,$

$43981, 44635, 47488]$

- The size 10 example illustrates the problems inherent with searching for a solution.

3. Partial History.

- Euler

- Prouhet (1851)

- Tarry (1910) - Small Examples

- Escott (1910) - Small Examples

- Letac (1941) - Size 9 and 10

- Gloden (1946) - Size 9 and 10

- Smyth (Math Comp. 1991) - Size 10 generalized.

4. Diophantine Form

Find distinct integers

$[\alpha_1, \ldots, \alpha_N]$ and $[\beta_1, \ldots, \beta_N]$

so that

$$\begin{aligned}
&\alpha_1 + \ldots + \alpha_N = \beta_1 + \ldots + \beta_n \\
&\alpha_1^2 + \ldots + \alpha_N^2 = \beta_1^2 + \ldots + \beta_n^2 \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\alpha_1^{N-1} + \ldots + \alpha_N^{N-1} = \beta_1^{N-1} + \ldots + \beta_N^{N-1}
\end{aligned}$$

- The problem is completely open for $N \geq 11$.

D. The Weak Prouhet-Tarry-Escott Problem.

1. Problem. For fixed N find $p \in Z[x]$

$$p(x) = (x-1)^N q(x) = \Sigma a_k x^k$$

that minimizes

$$S(p) = \Sigma |a_i|$$

or

$$S^2(p) = (\Sigma |a_i|^2)^{1/2}$$

2. Solving $S(p) = |S^2(p)|^2 = 2N$ is the Prouhet-Tarry-Escott-Problem and is the big prize.

3. Showing that there exist

$$\{p_N\} = \{(x-1)^N q(x)\}$$

so that

$$S(p_N) = o(N \log N)$$

is also a big prize.

• This shows that the "Easier Waring Problem" is easier than the "Waring Problem" (At the moment.)

• That is: it requires essentially fewer powers to write every integer as sums and differences of Nth powers than just as sums of Nth powers. (Fuchs and Wright, Quart. J. Math. 1936).

4. It is known that

$$S((x-1)^N q(x)) \leq \frac{N^2}{2}$$

is possible.

Any improvement would be a major step.

5. If we demand that p has a zero of order N but not $N+1$ at 1 then

$$S(p) = 0((\log N)N^2)$$

is possible (Hua).

Any improvement would be interesting.

E. Problem of Erdös and Szekeres (1958).

1. Problem. Minimize over $\{\alpha_1, \dots, \alpha_N\}$

$$S\left(\prod_{k=1}^{N}(1-x^{\alpha_i})\right)$$

Call this minimum S_N^π.

2. Conjecture. $S_N^\pi \gg N^k$ for any k.

3. From the P-T-E problem

$$S_N^\pi \geq 2N$$

4. Examples.

N	$\|f\|_1$	$\{\alpha_1, \ldots, \alpha_N\}$
1	2	$\{1\}$
2	4	$\{1, 2\}$
3	6	$\{1, 2, 3\}$
4	8	$\{1, 2, 3, 4\}$
5	10	$\{1, 2, 3, 5, 7\}$
6	12	$\{1, 1, 2, 3, 4, 5\}$
7	16	$\{1, 2, 3, 4, 5, 7, 11\}$
8	16	$\{1, 2, 3, 5, 7, 8, 11, 13\}$
9	20	$\{1, 2, 3, 4, 5, 7, 9, 11, 13\}$
10	24	$\{1, 2, 3, 4, 5, 7, 9, 11, 13, 17\}$
11	28	$\{1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19\}$
12	36	$\{1, \ldots, 9, 11, 13, 17\}$
13	48	$\{1, \ldots, 9, 11, 13, 17, 19\}$

5. Conjecture. Except for $N = 1, 2, 3, 4, 5, 6$ and 8

$$S_N^\pi \geq 2N + 2.$$

- Maltby solves this for N=7, 9 and 10.

6. Partials.

$S_N^\pi \ll N^{0(N^{1/2})}$ (Atkinson, Dobrowolski)

$S_N^\pi \ll N^{0(\log N N^{1/3})}$ (Odlyzko)

(could equally well use $\| \ \|_{L^2(D)}$.)

F. Lehmer's Conjecture.

Mahler's Measure: if

$$p(z) = \prod_{i=1}^{n} (z - \alpha_i)$$

then

$$M(p) = \prod_{i=1}^{n} \max\{1, |\alpha_i|\}$$

or equivalently

$$M(p) := \exp\left\{ \int_0^1 \log |p(e^{2\pi i t})| \, dt \right\}$$

Conjecture. Suppose p is a monic polynomial with integer coefficients. Then either $M(p) = 1$ or $M(p) > 1.17\ldots$.

• This generalizes Kronecker's theorem which can be stated as: $M(p) = 1$ implies that p is cyclotomic.

• The minimal Mahler measure for a non-cylotomic p is speculated to be

$$p := x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$$

for which $M(p) = 1.17628081825991750...$

• This is also speculated to be the smallest Salem number.

Question. Do there polynomials with coefficients $\{0, -1, +1\}$ with roots of arbitrarily high multiplicity inside the unit disk.

A negative answer solves the conjecture.

G. Littlewood's Conjecture.

Conjecture (resolved 1981). Suppose

$$p(z) := \sum_{n=0}^{N} c_n z^{K_n} \qquad \|c_i\| \geq 1.$$

Then the L_1 norm of p on the boundary of the unit disc is $>> \log(N)$.

H. A Problem of Erdős.

Conjecture, 1957. Suppose

$$p(z) := \sum_{n=0}^{N} c_n z^n \qquad c_i \pm 1.$$

Then the supremum norm of p on the boundary of the unit disc is $> (1+\epsilon)\sqrt{N}$.

I. Littlewood's Other Conjecture.

Conjecture (1966). There is some

$$p(z) := \sum_{n=0}^{N} c_n z^n \qquad c_i \pm 1$$

so that for all z on the boundary of the unit disc

$$C_1 < \frac{|p(z)|}{\sqrt{n}} < C_2.$$

• Littlewood, in part, based his conjecture on computations of all such polynomials up to degree twenty.

• Odlyzko has now done 200 MIPS years of computing on this problem

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SO MUCH FOR MOTIVATION

ON THE NUMBER OF ZEROS OF $\{0, +1, -1\}$ POLYNOMIALS AND RELATED CHEBYSHEV PROBLEMS.

P. Borwein, T. Erdélyi and G. Kós

• We consider the problem of minimizing the uniform norm on $[0,1]$ over polynomials p

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}$$

with fixed $|a_m| \neq 0$.

• This is equivalent to the question of how many zeros such a polynomial can have at 1.

• Particular cases include:

Polynomials with coefficients in the set $\{-1, 0, 1\}$.

Polynomials with coefficients in the set $\{0, 1\}$ on the interval $[-1, 0]$.

$$\mathcal{P}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R} \right\}$$

$$\mathcal{Z}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z} \right\}$$

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 0, 1\} \right\}$$

$$\mathcal{A}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{0, 1\} \right\}$$

So obviously

$$\mathcal{A}_n \subset \mathcal{F}_n \subset \mathcal{Z}_n \subset \mathcal{P}_n.$$

2. Number of Zeros at 1

Theorem 2.1. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1, \quad a_j \in \mathbb{C}$$

has at most

$$c \left(n(1 - \log |a_0|) \right)^{1/2}$$

zeros at 1.

• Applying Theorem 2.1 with $q(x) := x^{-n} p(x^{-1})$ gives the following:

Theorem 2.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}$$

has at most

$$c\,(n(1 - \log|a_n|))^{1/2}$$

zeros at 1.

• This sharpens a sequence of old and not so old results of Littlewood, Schur, Turán, Erdős, Bombieri, Vaaler and others. (In the case of small height polynomials.)

• The result is sharp.

Theorem 2.3. *It* $\exp(-3n) \le |a_0| \le 1$, *then there exists a polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j , \qquad |a_j| \le 1 , \quad a_j \in \mathbb{C}$$

such that p *has a zero at* 1 *with multiplicity at least*

$$\frac{1}{5}(n\,(1 - \log|a_0|))^{1/2} - 1 .$$

• The next two theorems treat the case $a_0 = 1$. The proofs are attractive and we will work through them. (As time allows.)

Theorem 2.4. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| = 1\,, \qquad |a_j| \le 1$$

has at most $5\sqrt{n}$ zeros at 1 and at most $c\sqrt{n}$ zeros in $[-1, 1]$.

Theorem 2.5. *For every $n \in \mathbb{N}$, there exists*

$$p_n(x) = \sum_{j=0}^{n^2} a_j x^j$$

such that $a_0 = 1$; $a_1, a_2, \ldots, a_{n^2}$ are real numbers of modulus less than 1; and p_n has a zero at 1 with multiplicity at least $n - 1$.

• Theorem 2.5 immediately implies

Corollary 2.6. *For every* $n \in \mathbb{N}$*, there exists a polynomial*

$$p_n(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad a_n = 1\,,$$

$a_0, a_1, \ldots, a_n$ *are real numbers of modulus less than* 1*, and* p_n *has a zero at* 1 *with multiplicity at least* $\lfloor \sqrt{n} \rfloor - 1$.

• The next related result is well known:

Theorem 2.7. *There is an absolute constant $c > 0$ so that for every $n \in \mathbb{N}$ there is a $p \in \mathcal{F}_n$ having at least $c\sqrt{n/\log(n+1)}$ zeros at* 1*.*

• Theorems 2.4 and 2.7 show that the right upper bound for the number of zeros a polynomial $p \in \mathcal{F}_n$ can have at 1 is somewhere between $c_1\sqrt{n/\log(n+1)}$ and $c_2\sqrt{n}$ with absolute constants $c_1 > 0$ and $c_2 > 0$.

• This gap looks quite hard to close. This is an old problem on which there has been no progress in 20 years.

• There is a simple observation about the maximal number of zeros a polynomial $p \in \mathcal{A}_n$ can have.

Theorem 2.8. *There is an absolute constant $c > 0$ such that every $p \in \mathcal{A}_n$ has at most $c \log n$ zeros at -1.*

• There is a less simple observation about the maximal number of zeros at 1 of a polynomial with coefficients $\{+1, -1\}$.

There are between $c_1 \log n$ and $c_2(\log n)^2$ such zeros and is open as to what is correct (Boyd 95).

Remark to Theorem 2.8. Let R_n be defined by

$$R_n(x) := \prod_{i=1}^{n} (1 + x^{a_i}),$$

where $a_1 := 1$ and a_{i+1} is the smallest odd integer that is greater than $\sum_{k=1}^{i} a_k$.

• It is tempting to speculate that R_n is the lowest degree polynomial with coefficients $\{0, 1\}$ and a zero of order n at -1.

• This is true for $n := 1, 2, 3, 4, 5$ but fails for $n := 6$ and hence for all larger n.

3. Restricted Chebyshev Problem

Theorem 3.1. *There are absolute constants so that*

$$\exp\left(-c_1 n(1-\log|a_m|))^{1/2}\right)$$

$$\leq \inf_p \|p\|_{[0,1]}$$

$$\leq \exp\left(-c_2 n(1-\log|a_m|))^{1/2}\right),$$

where the inf is taken over $0 \neq p$ *of the form*

$$p(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}$$

with $|a_m| \geq \exp\left(\frac{1}{2}(1-n)\right)$.

- This specializes to

Theorem 3.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_p \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right) ,$$

for polynomials of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j , \qquad |a_j| \leq 1 , \quad a_n = 1 .$$

• For the class $\mathcal{F}_n$ we have

Theorem 3.3. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right)$$

$$\leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]}$$

$$\leq \exp\left(-c_2\sqrt{n}\right).$$

• The upper bound requires some new ideas.

• The approximation rate in Theorems 3.2 and 3.3 should be compared with

$$\min_{p(x):=x^n+\cdots\in\mathcal{P}_n} \|p\|_{[0,1]}^{1/n} = \frac{2^{1/n}}{4}\,,$$

and also with

$$\frac{1}{2.376\ldots} < \min_{0\neq p\in\mathcal{Z}_n} \|p\|_{[0,1]}^{1/n} < \frac{1+o(1)}{2.3605}\,.$$

• The first equality above is attained by the normalized Chebyshev polynomial shifted linearly to $[0,1]$ and is proved by a simple perturbation argument. The second inequality is much harder (the exact result is open).

• It is an interesting fact that the polynomials $0 \neq p \in \mathcal{Z}_n$ with the smallest uniform norm on $[0, 1]$ are very different from the usual Chebyshev polynomial of degree n.

• For example, they have at least 52% of their zeros at either 0 or 1. Relaxation techniques do not allow for their approximate computation.

• Likewise, polynomials $0 \neq p \in \mathcal{F}_n$ with small uniform norm on $[0, 1]$ are again quite different from polynomials $0 \neq p \in \mathcal{Z}_n$ with small uniform norm on $[0, 1]$.

• The story is roughly as follows. Polynomials $0 \neq p \in \mathcal{P}_n$ with leading coefficient 1 and with smallest possible uniform norm on $[0, 1]$ are characterized by equioscillation and are given by the Chebyshev polynomials explicitly.

• In contrast, finding polynomials from $\mathcal{Z}_n$ with small uniform norm on $[0, 1]$ is closely related to finding irreducible polynomials with all their roots in $[0, 1]$.

• As we shall see the construction of small norm polynomials from $\mathcal{F}_n$ is governed by how many zeros such a polynomial can have at 1.

• It is interesting to note that the polynomials $0 \neq p \in \mathcal{P}_n$ with leading coefficient 1 and with smallest uniform norm on $[0, 1]$ have coefficients that alternate in sign.

• This also appears to be true for the analogous polynomials from $\mathcal{Z}_n$ (though this is only conjectural and probably quite hard to prove).

• This is quite different from the story for $\mathcal{F}_n$. For polynomials $p(-x)$ with $0 \neq p \in \mathcal{A}_n$ we get a very much larger smallest possible uniform norm on $[0,1]$.

Theorem 3.4. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1 \log^2(n+1)\right)$$

$$\leq \inf_{0 \neq p \in \mathcal{A}_n} \|p(-x)\|_{[0,1]}$$

$$\leq \exp\left(-c_2 \log^2(n+1)\right)$$

4. Tools

• In the general case the tools are:

• Denote by $\mathcal{S}$ the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \leq \frac{1}{1-|z|}\,, \qquad z \in D\,.$$

Theorem 4.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$|f(0)|^{c_1/a} \leq \exp\left(\frac{c_2}{a}\right) \|f\|_{[1-a,1]}$$

for every $f \in \mathcal{S}$ and $a \in (0,1]$.

Hadamard Three Circles Theorem.

Suppose f is regular. Let $M(r) := \max_{|z|=r} |f(z)|$. Then for $r_1 < r < r_2$

$$M(r)^{\log(r_2/r_1)} \le M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}.$$

Halász Lemma.

For every $k \in \mathbb{N}$, there exists a polynomial $h \in \mathcal{P}_k^c$ such that

$$h(0) = 1\,, \quad h(1) = 0\,, \quad |h(z)| < \exp\left(\frac{2}{k}\right)$$

for $|z| \le 1$.

5. Proofs of the Main Results

Theorem 2.4. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_n| = 1, \qquad |a_j| \leq 1$$

has at most $5\sqrt{n}$ zeros at 1.

Proof of Theorem 2.4. If p has a zero at 1 of multiplicity m, then for every polynomial f of degree less than m, we have
(*)

$$a_0 f(0) + a_1 f(1) + \cdots + a_n f(n) = 0\,.$$

We construct a polynomial f of degree at most $5\sqrt{n}$, for which

$$f(n) > |f(0)| + |f(1)| + \cdots + |f(n-1)|\,.$$

Equality (*) cannot hold with this f, so the multiplicity of the zero of p at 1 is at most the degree of f.

Let T_ν be the ν-th Chebyshev poly. Let

$$g := T_0 + T_1 + \cdots + T_k \in \mathcal{P}_k\,.$$

Note that $g(1) = k + 1$ and

$$g(\cos y) = 1 + \cos y + \cos 2y + \cdots + \cos ky$$

$$= \frac{\sin(k+\frac{1}{2})y + \sin\frac{1}{2}y}{2\sin\frac{1}{2}y}\,.$$

Hence, for $-1 \le x < 1$,

$$|g(x)| \le \frac{\sqrt{2}}{\sqrt{1-x}} \,.$$

Let $f(x) := g^4(\frac{2x}{n} - 1)$. Then $f(n) = (k+1)^4$ and

$$|f(0)|+|f(1)|+\cdots+|f(n-1)| \le \sum_{j=1}^{n} \frac{4}{\left(\frac{2j}{n}\right)^2} < \frac{\pi^2}{6} n^2 \,.$$

If $k := \lfloor (\pi^2/6)^{1/4} \sqrt{n} \rfloor$ then

$$f(n) > |f(0)| + |f(1)| + \cdots + |f(n-1)| \,.$$

In this case the degree of f is $4k \le 5\sqrt{n}$. $\square$

Theorem 2.5. *For every $n \in \mathbb{N}$, there exists*

$$p_n(x) = \sum_{j=0}^{2n^2} a_j x^j$$

such that $a_{2n^2} = 1$; $a_0, a_1, \ldots, a_{2n^2-1}$ are real numbers of modulus less than 1; and p_n has a zero at 1 with multiplicity at least n.

Proof of Theorem 2.5. Define

$$L_n(x) := \frac{(n!)^2}{2\pi i} \int_\Gamma \frac{x^t\, dt}{\prod_{k=0}^{n} (t - k^2)}$$

where the simple closed contour Γ surrounds the zeros of the denominator in the integrand.

Then L_n is a polynomial of degree n^2 with a zero of order n at 1.

Also, by the residue theorem,

$$L_n(x) = 1 + \sum_{k=1}^{n} c_{k,n} x^{k^2}$$

where

$$c_{k,n} = \frac{(-1)^n (n!)^2}{\prod_{j=0, j\neq k}^{n} (k^2 - j^2)} = \frac{(-1)^k 2(n!)^2}{(n-k)!(n+k)!}$$

It follows that

$$c_{k,n} \leq 2\,, \qquad k = 1, 2, \ldots, n$$

Hence,

$$q_n(x) := \frac{L_n(x) + L_n(x^2)}{2}$$

is a polynomial of degree $2n^2$ with real coefficients and with a zero at 1 of order n.

Also q_n has constant coefficient 1 and each of its remaining coefficients is a real number of modulus less than 1.

Now let $p_n(x) := x^{2n^2} q_n(1/x)$. $\Box$

Proof of Theorem 2.8. Suppose $P \in \mathcal{A}_n$ has m zeros at -1. Then $(1+x)^m$ divides P. On evaluating the above at 1 we see that $n \geq 2^m - 1$ and the result follows. $\Box$

6. Comments

- There is an obvious interval dependence in the problem of minimal elements from $\mathcal{F}_n$.

- On any interval $[0, \delta]$ with $\delta < 1/2$ the only polynomials from $\mathcal{F}_n$ with minimal uniform norm are $\pm x^n$.

- On $[0, 1/2]$ all of $\pm x^n$ and $\pm(x^n - x^{n-1})$ are extremals.

- On any interval $[0, \delta]$ with $\delta > 1/2$ the polynomials $\pm(x^n - x^{n-1})$ work better than x^n, so the nature of the extremals change at 1/2.

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