INCOMPLETE RATIONALS

Peter Borwein
Simon Fraser University

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Introduction:

We consider rational approximations of the form

$$\left\{ (1+z)^{\alpha n+1} \frac{p_{cn}(z)}{q_n(z)} \right\}$$

in certain natural regions in the complex plane where p_{cn} and q_n are polynomials of degree cn and n respectively.

In particular we construct natural maximal regions (as a function of α and c) where the collection of such rational functions is dense in the analytic functions.

So from this point of view we have rather complete analogue theorems to the results concerning incomplete polynomials on an interval.

The analysis depends on a careful examination of the zeros and poles of the Padé approximants to $(1+z)^{\alpha n+1}$. This is effected by an asymptotic analysis of certain integrals.

In this sense it mirrors the well known results of Saff and Varga on the zeros and poles of the Padé approximant to exp. Results that, in large measure, we recover as a limiting case.

In order to make the asymptotic analysis as painless as possible we prove a fairly general result on the behavior, in n, of integrals of the form

$$\int_0^1 [t(1-t)f_z(t)]^n dt$$

where $f_z(t)$ is analytic in z and a polynomial in t.

From this we can and do analyze automatically (by computer) the limit curves and regions that we need.

The Wellspring:

In a remarkable paper of 1924, Szegő considered the zeros of the partial sums $s_n(z) := \sum_{k=0}^{n} z^k/k!$ of the MacLaurin expansion for e^z. Szegő established that $\hat{z}$ is a limit point of zeros of the sequence of normalized partial sums,

$$\{s_n(nz)\}_{n=0}^{\infty},$$

if and only if

$$\hat{z} \in \{z : |ze^{1-z}| = 1, |z| \leq 1\}.$$

Moreover, Szegő showed that $\hat{z}$ is a nontrivial limit point of zeros of the normalized remainder

$$\{e^{nz} - s_n(nz)\}_{n=1}^{+\infty}$$

if and only if

$$\hat{z} \in \{z : |ze^{1-z}| = 1, |z| \geq 1\}.$$

Padé Approximation to $(1+z)^{\alpha n}$:

Theorem. *The set of functions*

$$\{(1+z)^{\alpha n} r_n(z) : r_n(z) \in \pi_{n,n}\}_{n=1}^{+\infty}$$

is dense in $A(K)$, the analytic functions on K, where K is an arbitrary compact subset of R_3 and not in any region strictly containing R_3 (R_3 is the region in Fig.1).

Theorem 2.1. *For the (m,n) Padé approximation to $(1+z)^{\alpha n+1}$ at 0, $\alpha > 0$, we have*

(a)
$$(1+z)^{\alpha n+1} - \frac{p_m(z)}{q_n(z)}$$
$$= \frac{z^{m+n+1}\int_0^1 (1-t)^m t^n (1+tz)^{\alpha n - m} dt}{q_n(z)},$$

(b)
$$p_m(z) = \int_0^1 (t-1)^n t^{\alpha n - m} (1+z-t)^m dt,$$

and

(c)
$$q_n(z) = \int_0^1 (1-t)^m t^{\alpha n - m} (t(z+1)-1)^n dt.$$

Corollary 2.2. *For the (cn, n) Padé approximation to $(1+z)^{\alpha n+1}$ at 0, $\alpha > 0$, we have*

(a)
$$(1+z)^{\alpha n+1} - \frac{p_{cn}(z)}{q_n(z)}$$

$$= \frac{z^{cn+n+1}\int_0^1 [(1-t)^c t(1+tz)^{\alpha-c}]^n dt}{q_n(z)},$$

(b)
$$p_{cn}(z) = \int_0^1 [(t-1)t^{\alpha-c}(1+z-t)^c]^n dt,$$

and

(c)
$$q_n(z) = \int_0^1 [(1-t)^c t^{\alpha-c}(t(1+z)-1)]^n dt.$$

Corollary 2.3. *When $c = 1$, we have*

$$(1+z)^n p_n\left(\frac{-z}{1+z}\right) = q_n(z).$$

The Asymptotic Analysis

Theorem 2.4. *Let*

$$I_n = \int_0^1 [t(1-t)f(t)]^n dt = \int_0^1 [Q(t)]^n dt$$

where $Q(t) = t(1-t)f(t)$ *is a polynomial of degree* N *in* t.

Let $t_1, t_2, \ldots, t_{N-1}$ *be the* $N-1$ *zeros of* $Q'(t)$.

Suppose that

$$|Q(t_i)| \neq |Q(t_j)|, \qquad i \neq j.$$

Then

$$\lim_{n\to\infty} I_n^{1/n} = arg(Q(t_i))|Q(t_i)| = Q(t_i)$$

for some i.

Theorem 2.5. *Let*

$$I_n(z) = \int_0^1 [t(1-t)f_z(t)]^n dt = \int_0^1 [Q_z(t)]^n dt$$

where $Q_z(t) = t(1-t)f_z(t)$ *is a polynomial in* t *and analytic in* z *on an open connected set* U*. Suppose*

$$|Q_z(t_i(z))| \neq |Q_z(t_j(z))|$$

for any $i \neq j$*, and any* $z \in U$*, where* $t_i := t_i(z)$ *are the zeros of the polynomial* $\frac{d}{dt}Q_z(t)$ *(which by the above assumption can be given so that each* t_i *is analytic on* U*). Then*

(a) $I_n(z)^{1/n}$ *converges to a non-zero limit pointwise on* U*.*

(b) $|I_n(z)|^{1/n}$ *is uniformly bounded on compact subsets of* U*.*

(c) $I_n(z)^{1/n}$ *converges uniformly to a* $Q_z(t_i(z))$ *on compact subsets of* U*, and* $Q_z(t_i(z))$ *is analytic on* U*. Moreover,* $Q_z(t_i(z)) \neq 0$ *for all* $z \in U$ *.*

Corollary 2.6. *Let $I_n(z)$, $f_z(t)$ and $Q_z(t)$ be as in Theorem 2.5. Suppose that for each z, $Q_z(t)$ is a polynomial of degree N in t, and further that $Q_z(t)$ is analytic in z. Then, the limit points of the zeros of $I_n(z)$ can only cluster on the curve*

$$\{z : |Q_z(t_i(z))| = |Q_z(t_j(z))|, \qquad \text{for some } i \neq j\}$$

or at points where $Q_z(t_i(z)) = 0$, or at points where $Q_z(t_i(z))$ is not analytic. (Note $t_i := t_i(z)$, which is a function of z.)

These results require some careful saddle point analysis.

The nice thing is that one can guarantee that the right contours exist without actually constructing them.

Computing the possible limit curves is now an exercise in computer algebra.

Specialization to $c = 1$

$$p_n(z) = \int_0^1 [(t-1)t^{\alpha-1}(1-t+z)]^n dt,$$
$$q_n(z) = \int_0^1 [(1-t)t^{\alpha-1}(t(1+z)-1)]^n dt,$$

$$e_{(\alpha-1)n}(z) = \int_0^1 [(1-t)t(1+tz)^{\alpha-1}]^n dt.$$

Let $Q_z(t) = (1-t)t^{\alpha-1}(t(1+z)-1)$, then

$$Q_z(0) = Q_z(1) = Q_z\left(\frac{1}{1+z}\right) = 0,$$

and

$$\frac{d}{dt}Q_z(t)\,\Big|_{t=t_{1,2}(z)} = 0$$

where

$$t_{1,2}(z) = \frac{\alpha(z+2) \pm \mu}{2(z+1)(1+\alpha)},$$

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

Therefore, from Corollary 2.6 and the above observation, the critical curve for $p_n(z)$, $q_n(z)$ and $e_{(\alpha-1)n}(z)$ is

$$\{z : |Q_z(t_1(z))| = |Q_z(t_2(z))|\},$$

which is

$$\left\{ \left|\frac{\alpha z + 2z + 2 + \mu}{\alpha z + 2z + 2 - \mu}\right| \left|\frac{\alpha z - 2 - \mu}{\alpha z - 2 + \mu}\right| \left|\frac{\alpha z + 2\alpha - \mu}{\alpha z + 2\alpha + \mu}\right|^{\alpha-1} = 1 \right\}$$

where

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

The critical curves for $\alpha = 2$, $\alpha = 3$, $\alpha = 5$ and $\alpha = 8$ are shown in Figures 1, 2, 3 and 4 respectively.

In Figures 5 and 6, we plot the zeros of $p_n(z)$ and $q_n(z)$ for $\alpha = 2$, $n = 20$ and $\alpha = 3$, $n = 10$ respectively. We also plot the zeros of $e_{(\alpha-1)n}(z)$ for $\alpha = 3$, $n = 15$ in Figure 7.

Theorem 3.1. *For $\alpha > 1$, $\{q_n(z)\}^{1/n}$ converges to $Q_z(t_2(z))$ uniformly on any compact subset of R_1, R_2 and R_3, and to $Q_z(t_1(z))$ uniformly on any compact subset of R_4. Moreover, the limit points of the zeros of $\{q_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_3, which is the boundary between R_3 and R_4.*

Theorem 3.2. *For $\alpha > 1$, $\{p_n(z)\}^{1/n}$ converges to $(1+z)^{\alpha} Q_z(t_1(z))$ uniformly on any compact subset of R_1 and R_2, and to $(1+z)^{\alpha} Q_z(t_2(z))$ uniformly on any compact subset of R_3 and R_4. Moreover, the limit points of the zeros of $\{p_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_2, which is the boundary between R_2 and R_3.*

Theorem 3.3. *For $\alpha > 1$, $\{e_{(\alpha-1)n}\}^{1/n}$ converges to $(1+z)^{\alpha} Q_z(t_2(z))/z^2$ uniformly on any compact subset of R_1 and to $(1+z)^{\alpha} \varphi_z(t_1(z))/z^2$ uniformly on any compact subset of R_2, R_3 and R_4. Moreover, the limit points of the zeros of $\{e_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_1, which is the boundary between R_1 and R_2.*

Theorem 4.1. *Let $p_n(z)$, $q_n(z)$ and $e_{(\alpha-1)n}(z)$ be as in Corollary 2.2 in the case $c = 1$. Then we have that $(1+z)^{\alpha n+1} q_n(z)/p_n(z)$ converges*

(a) *to ∞ uniformly on any compact subset of R_1 and R_4 (as in Figures 1,2,3,4);*

(b) *to 0 uniformly on any compact subset of R_2;*

(c) *to 1 uniformly on any compact subset of R_3.*

Remark. *Observe that 1 can not be approximated on any region strictly larger than R_3 by the Rouché's Theorem, so R_3 is a natural maximal region of denseness.*

Theorem 4.2.

$$\{(1+z)^{\alpha n} r_n(z) : r_n(z) \in \pi_{n,n}\}_{n=1}^{\infty}$$

is dense in $A(K)$ where K is an arbitrary compact subset of R_3.

The Polynomial Case: c=0

Theorem 5.1. *If c:=0 then* $q_n(z)$ *and* $e_{\alpha n}(z)$ *have the same critical curve*

$$\{z : |z(1+z)^{\alpha}| = \alpha^{\alpha}/(1+\alpha)^{1+\alpha}\}.$$

and the limit points of the zeros of $q_n(z)$ *or* $e_{\alpha n}(z)$ *can only cluster on this curve .*

Theorem 5.4.

$$\{(1+z)^{\alpha n} p_n(z) : p_n(z) \in \pi_n\}_{n=1}^{+\infty}$$

is dense in $A(K)$ *where* K *is an arbitrary compact subset of* R_3 *as in Fig. 8.*

The limit points of the zeros of $\{q_n(z)\}_{n=1}^{\infty}$ *are dense in the boundary between* R_1 *and* R_3*.*

The limit points of the zeros of $\{e_{\alpha n}(z)\}_{n=1}^{\infty}$ *are dense in the boundary between* R_1 *and* R_2*.*

Question. For which sets K is

$$\{(1+z)^{\alpha n} r_n(z)\}_{n=1}^{\infty}$$

dense in $A(K)$?

Uniform convergence to 1 of $\{x^{\theta n} r_n(x)\}$ is not possible on any interval $[b, 1]$ with

$$b < \tan^4(\pi(\theta - 1)/4\theta)$$

and this is essentially sharp. (Saff and Rachmanov)

Uniform convergence to 1 of $\{e^x r_n(x)\}$ is not possible on any interval $[0, a]$ with $a > 2\pi$ (compare $b = 2$ for polynomials).

Question. What is the maximum measure of

$$\{z : |r_n'(z)/r_n(z)| >= n\}$$

in the complex plane? (On the line it should be 2π ?)