INTEGER CHEBYSHEV POLYNOMIALS

Peter Borwein

Simon Fraser University

1. Introduction.

The ubiquitous Chebyshev polynomial

$$T_n(x) := \cos(n \arccos x)$$

$$= \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x+\sqrt{x^2-1}\right)^n\right]$$

is a polynomial of degree n with integer coefficients and with lead coefficient 2^{n-1} that equioscillates $n+1$ times on the interval $[-1,1]$.

For fairly simple reasons, based on this equioscillation, it follows that

$$\min_{p_{n-1}\in\mathcal{P}_{n-1}} \|x^n - p_{n-1}\|_{[-1,1]}$$

$$= \left\|2^{1-n}T_n\right\|_{[-1,1]} = \frac{1}{2^{n-1}}.$$

Here $\mathcal{P}_n$ denotes the polynomials of degree at most n with real coefficient.

The polynomial

$$p_n(x) := 2\left(\frac{b-a}{4}\right)^n T_n\left(\frac{2x-a-b}{b-a}\right)$$

is now the monic polynomial of degree n of **smallest supremum norm** on the interval $[a,b]$ and it satisfies

$$\|p_n\|_{[a,b]} = 2\left(\frac{b-a}{4}\right)^n.$$

So

$$(\|p_n\|_{[a,b]})^{1/n} \to \left(\frac{b-a}{4}\right)$$

which gives the **transfinite diameter** of $[a,b]$.

The Chebyshev polynomials have a central role to play in minimization problems in the sup norm as well as many other extremal problems .

The analogue problem where the polynomials are restricted to have integer coefficients is very much harder.

We let $\mathcal{Z}_n$ denote the polynomials of degree at most n with integer coefficients.

We define

$$(*)\qquad \Omega_n[a,b] := \left(\inf_{0 \neq p \in \mathcal{Z}_n} \|p\|_{[a,b]} \right)^{1/n}$$

and let

$$\Omega[a,b] := \inf\{\Omega_n[a,b] : n = 0, 1, \dots\}$$

$$= \lim_{n \to \infty} \Omega_n[a,b].$$

Any polynomial satisfying $(*)$ above is called an n-th **integer Chebyshev polynomial** on $[a,b]$.

The above limit exists and equals the inf mostly because

$$(\Omega_{n+m}[a,b])^{n+m} \leq (\Omega_n[a,b])^n \, (\Omega_m[a,b))^m \, .$$

We have from the unrestricted case the trivial inequality

$$\Omega[a,b] \geq \frac{b-a}{4}.$$

We also have

$$\Omega[a,b] \leq \Omega_n[a,b]$$

for any particular n.

Thus good upper bounds can be achieved by computation (although the computation to any degree of accuracy is hard).

The limit $\Omega[a,b]$ may be thought an integer version of the **transfinite diameter**.

Hilbert showed that there exists an absolute constant c so that

$$\inf_{0\neq p\in\mathcal{Z}_n} \|p\|_{L_2[a,b]} \leq cn^{1/2}\left(\frac{b-a}{4}\right)^{1/2}$$

and Fekete showed that

$$(\Omega_n[a,b])^n \leq 2^{1-2^{-n-1}}(n-1)\left(\frac{b-a}{4}\right)^{n/2}$$

There are many refinements.

From the above it follows that

$$\frac{b-a}{4} \leq \Omega[a,b] \leq \left(\frac{b-a}{4}\right)^{1/2}$$

Recall that $b-a\leq 4$.

There is a pretty argument due to Gelfond to see that integer coefficients really are a restriction on $[0,1]$.

If $0 \neq p_n \in \mathcal{Z}_n$ then

$$\|p_n\|^2_{[0,1]} \geq \|p_n\|^2_{L_2[0,1]} = \int_0^1 p_n^2(x)dx$$

$$= \frac{m}{\mathrm{LCM}(1, 2, \dots, 2n+1)} \neq 0$$

where LCM denotes the least common multiple. Now $\mathrm{LCM}(1, 2, \dots, n))^{1/n} \sim e$, by the prime number theorem and it follows that

$$\Omega[0,1] \geq 1/e.$$

This is not however the right lower bound.

The best previous bounds, due to Aparicio and Gorshkov,

$$\frac{1}{(2.37686\dots)} \leq \Omega[0,1] \leq \frac{1}{(2.343\dots)}.$$

The upper bound comes by example.

For the lower bound we show there exist infinitely many relatively prime polynomials $q_k \in \mathcal{Z}_k$ with all their roots in $(0,1)$, and with lead coefficients $\{a_k\}$ satisfying $a_k^{1/k} \leq 2.37686\ldots$

This number was conjectured to be the right bound. It comes from iterating $(x - 1/x)$ on $(-\infty, \infty)$.

Lemma. *Suppose $p_n \in \mathcal{Z}_n$ and suppose $q_k(z) := a_k z^k + \cdots + a_0 \in \mathcal{Z}_k$ has all its roots in $[a,b]$. If p_n and q_k do not have common factors then*

$$\left(\|p_n\|_{[a,b]}\right)^{1/n} \geq |a_k|^{-1/k}.$$

Proof. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the roots of q_k. Then

$$|a_k|^n p_n(\beta_1) p_n(\beta_2) \cdots p_n(\beta_k)$$

is a non-zero integer and the result follows. $\square$

Aparicio also shows that if $[a,b]=[0,1]$ then any polynomial $p\in\mathcal{Z}_n$ for which the inf in (*) is achieved, for sufficiently large n, has a factor of the form

$$(x)^{\lfloor\lambda, n\rfloor}(1-x)^{\lfloor\lambda_1 n\rfloor}(2x-1)^{\lfloor\lambda_2 n\rfloor}$$

$$*(5x^2-5x+1)^{\lfloor\lambda_3 n\rfloor}$$

where $\lambda_1\geq .014$, $\lambda_2\geq .016$, $\lambda_3\geq .0037$.

We improve the upper bounds of $\Omega[0,1]$ to

$$\Omega[0,1]\leq\frac{1}{2.360}$$

and use this to increase the number of factors that must divide an n-th integer Chebyshev polynomial T_n on $[0,1]$.

We also establish a lower bound for the multiplicity of the zero at 0 of the integer Chebyshev polynomial $T_{n,a}$ on $[a,1]$.

From all this we deduce that the natural conjecture, due to the Chudnovsky's, that

$$\Omega[0,1] = \frac{1}{2.37686..}$$

is close but false.

These high order zeros are also used to show that the function $\Omega(x) := \Omega[0,x]$ is constant on the interval $[1-\delta, 1+\delta]$.

We also relate the integer Chebyshev problem on small intervals $[0, 1/m]$ to an old problem of Schur and Siegel on the trace of totally positive algebraic integers.

Another related problem is the Prouhet-Tarry-Escott problem.

We conclude with a number of open problems.

2. Computing Integer Chebyshev Polys.

We restrict our attention to the interval $[0,1]$. Though we observe in passing that

$$(\Omega[-1,1])^4 = (\Omega[0,1])^2 = \Omega[0,1/4]$$

as a consequence of the changes of variable $x \to x^2$ and $x \to x(1-x)$ and symmetry.

The dependence of the constant $\Omega[a,b]$ and the minimal polynomials on $[a,b]$ is interesting and is explored a little further later.

Even computing low degree examples is complicated. There is no good algorithm and getting examples of say degree 100 seems intractable.

n	n-th integer Chebyshev poly on [0,1]
1	x or $(1-x)$ or $(2x-1)$
2	$x(1-x)$
3	$x(1-x)(2x-1)$
4	$x^2(1-x)^2$ or $x(1-x)(2x-1)^2$
5	$x^2(1-x)^2(2x-1)$
6	$[x(1-x)(2x-1)]^2$

Note that we do not have uniqueness, though it is open as to whether we have uniqueness for n sufficiently large.

The arguments for the above table are of the following variety.

Consider the case $n = 5$.

Let $T_5 \in \mathcal{Z}_5$ be a 5-th integer Chebyshev polynomial on $[0, 1]$.

Then $T_5(0)$ and $T_5(1)$ are integers of modulus less then 1 so both of them must be 0.

Using Markov's inequality, we obtain that

$$\|T_5'\|_{[0,1]} \le 50\|T_5\|_{[0,1]} \le \frac{50}{(2.236\ldots)^5} < 1.$$

Since $T_5'(0)$ and $T_5'(1)$ are integers of modulus less than 1, both of them must be zero.

Since $2^5 T_5(1/2)$ is an integer of modulus at most $32/(2.236\ldots)^5 < 1$, it must also be zero.

Thus we conclude that

$$x^2(1-x)^2(2x-1) \ \text{divides} \ T_5.$$

Examples in $L_2[0,1]$. Polynomials of degrees $13\ldots,20$, which minimize $\|p_n\|_{L_2[0,1]}$ This computation was done in pari by using the minum function.

Degree 13

$$(5z^2-5z+1)(2z-1)^2z^4(z-1)^5$$

$$(5z^2-5z+1)(2z-1)^2(z-1)^4z^5$$

$$(5z^2-5z+1)(2z-1)^3(z-1)^4z^4$$

$$(2z-1)(5z^2-5z+1)(z-1)^5z^5$$

$$(2z-1)(5z^2-5z+1)^2(z-1)^4z^4$$

$$(2z-1)^3(z-1)^5z^5$$

$$(2z-1)(z-1)^4z^4(29z^4-58z^3+40z^2-11z+1)$$

Degree 14

$(5z^2-5z+1)(2z-1)^2(z-1)^5z^5$

Degree 15

$(5z^2-5z+1)(2z-1)^3(z-1)^5z^5$

Degree 16

$(5z^2-5z+1)(2z-1)^2(z-1)^6z^6$

Degree 17

$(5z^2-5z+1)(2z-1)^3(z-1)^6z^6$

Degree 18

$(2z-1)^2(5z^2-5z+1)(z-1)^6z^6$

Degree 19

$(5z^2 - 5z + 1)(2z - 1)^3(z - 1)^7 z^7$

Degree 20

$(5z^2 - 5z + 1)(29z^4 - 58z^3 + 40z^2 - 11z + 1)(2z - 1)^2(z - 1)^6 z^6$

Let

$$
\begin{aligned}
p_0(x) :=& x \\
p_1(x) :=& 1 - x \\
p_2(x) :=& 2x - 1 \\
p_3(x) :=& 5x^2 - 5x + 1 \\
p_4(x) :=& 13x^3 - 19x^2 + 8x - 1 \\
p_5(x) :=& 13x^3 - 20x^2 + 9x - 1 \\
p_6(x) :=& 29x^4 - 59x^3 + 40x^2 - 11x + 1 \\
p_7(x) :=& 31x^4 - 61x^3 + 41x^2 - 11x + 1 \\
p_8(x) :=& 31x^4 - 63x^3 + 44x^2 - 12x + 1 \\
p_9(x) :=& 941x^8 - 3764x^2 + 6349x^6 - 5873x^5 \\
& 3243x^4 - 1089x^3 + 216x^2 - 23x + 1
\end{aligned}
$$

We have

Proposition. *Let*

$$P_{210} := p_0^{67} \cdot p_1^{67} \cdot p_2^{24} \cdot p_3^{9} \cdot p_4 \cdot p_5 \cdot p_6^{3} \cdot p_7 \cdot p_8 \cdot p_9$$

then

$$\left(\|P_{210}\|_{[0,1]}\right)^{1/210} = \frac{1}{(2.3543\dots)}$$

and hence

$$\Omega[0,1] \leq \frac{1}{(2.3543\dots)}.$$

Proof. This proof is obviously just a computational verification. It is the algorithm for finding P_{210} which is of some interest. It is based on LLL lattice basis reduction in the following way.

a] Lattice basis reduction finds a short vector in a lattice. If we construct a lattice of the form

$$p(z) \cdot \sum_{k=0}^{n} \alpha_k z^k = \sum_{k=0}^{m} \beta_k z^k$$

where p is a fixed polynomial and the set

$$\{(\alpha_0, \alpha_1, \ldots, \alpha_n)\}$$

is a lattice then the set

$$\{(\beta_0, \beta_1, \ldots, \beta_m)\}$$

is also a lattice, and LLL will return a short vector in the sense of $\sum_{k=0}^{m} |\beta_k|^2$ being relatively small.

Observe that $\left(\sum_{k=0}^{m} |\beta_k|^2\right)^{1/2}$ is just the L_2 norm on the unit disk of the polynomial $\sum_{k=0}^{m} \beta_k z^k$.

So LLL lets us find polynomials of small L_2 norm (and hence small sup norm) on the disk, and we can do this while preserving divisibility by a fixed p.

b] Convert the problem from the interval $[\alpha, \beta]$ to the disk. This is easy. One first maps $[\alpha, \beta]$ to $[-2, 2]$ by a linear change of variables. One then lets $x := z + 1/z$. This maps a polynomial in x

on $[-2, 2]$ to a polynomial in z and $1/z$ on the boundary of the unit disk.

c] Attack the problem incrementally by using a] and b]. That is, at the k-th stage find a polynomial q_k of degree kN divisible by q_{k-1} of degree $(k-1)N$ using LLL on a lattice of size $N + 1$. This allows us to keep the size of LLL fairly small and uses the fact that integer Chebyshev polynomials tend to have (of necessity) many repeat factors. We used $N = 10$ in the actual computation and started with $q_0 := 1$. $\square$

We can computationally refine the above.

Proposition. *The inequality*

$$\Omega[0, 1/4] \leq \frac{1}{(5.5723\ldots)}$$

and hence

$$\Omega[0, 1] \leq \frac{1}{(2.3605\ldots)}.$$

holds.

This is done by minimizing over

$$P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_9^{\alpha_9}$$

which is a linear problem.

Corollary. *Let k be a positive integer, and let P_{210} be as in the previous Proposition. Then $(P_{210})^k$ divides all the n-th integer Chebyshev polynomials on $[0, 1]$ provided n is sufficiently large.*

Proof. Each p_i, $i = 0, 1, \cdots, 9$ is irreducible and satisfies

$$p_i(x) = a_k x^k + a^{k-1} x^{k-1} + \cdots + a_0$$

with

$$|a_k|^{1/k} < 2.36.$$

Each p_i also has all roots in $[0, 1]$. It follows now by the first Lemma that if Q is a polynomial of degree n with integer coefficients, and

$$\left(\|Q_n\|_{[0,1]}\right)^{1/n} \leq \frac{1}{2.3605}$$

then p_i divides Q.

Markov's inequality gives the arbitrarily high multiplicity eventually. $\square$

We deduce immediately as above.

Corollary. *The polynomials*

$$p_0, p_1, \cdots, p_9$$

are the only irreducible polynomials with all their roots in $[0, 1]$ *of the form*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

with

$$|a_n|^{1/n} < 2.36.$$

3. Finer Structure.

The exact dependence of $\Omega[a, b]$ on the interval $[a, b]$ is interesting and complicated. If we let

$$\Omega(x) := \Omega[0, x]$$

Then clearly Ω is a non-decreasing function on $(0, \infty)$. Obviously

$$\lim_{x \to 0} \Omega(x) = 0$$

(consider x^m on $[0, \delta]$). So $\Omega(x)$ maps $[0, 4]$ onto $[0, 1]$.

It is an exercise to show that Ω is in fact continuous. This follows mostly from a theorem of Chebyshev that gives

$$\|p_n\|_{[0,\delta+\epsilon]} \leq (1 + k_{\epsilon,\delta})^n \|p_n\|_{[0,\delta]}$$

for every $p_n \in \mathcal{P}_n$.

What is less obvious is that $\Omega(x)$ is locally flat on many intervals. Indeed it is conceivable that the derivative of Ω is almost everywhere zero.

Theorem. *Let* $T_n := T_n\{[0,1]\}$ *be an* n*-th integer Chebyshev polynomial on* $[0,1]$*. Then* T_n *is of the form*

$$T_n(x) = x^k(1-x)^k S_{n-2k}(x)$$

where $(0.26)n < k$ *if* n *is large enough.*

As a consequence, there exists an absolute constant $\delta > 0$ *(independent of* n*) so that* T_n *is an* n*-th integer Chebyshev polynomial on larger intervals* $[-a, 1+a]$ *for every* $a \in (0, \delta]$*.*

4. The Schur-Siegel Trace Problem.

Let $\alpha := \alpha_1$ be an algebraic number with conjugate roots $\alpha_2, \cdots, \alpha_n$. We say that α is **totally real (positive)** if all the α_i are real (positive). The **trace** of a totally positive algebraic integer is

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Except for finitely many explicit exceptions, if α is a **totally real algebraic integer** then then

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq 1.648, \quad \text{Schur (18)}$$

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq 1.733, \quad \text{Siegel (43)}$$

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq 1.771, \quad \text{Smyth (83).}$$

Note that $4\cos^2(\pi/p)$ is a totally positive algebraic integer of degree $(p-1)/2$ and trace $p-2$ for p prime. So the best constant in the above theorem is less than 2.

Connection to the integer Chebyshev Problem is

Proposition. *If*

$$\Omega[0, 1/m] < \frac{1}{m+\delta}.$$

Then, with finitely many exceptions,

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} \geq \delta$$

for every totally positive algebraic integer α_1 of degree $d > 1$ with conjugates $\alpha_2, \ldots, \alpha_d$.

Proof. Mostly an application of the original lemma and the Arithmetic-Geometric mean inequality. □

Corollary. *If α_1 is a totally positive algebraic integer of degree $d > 1$ with conjugates $\alpha_2, \ldots, \alpha_d$ then*

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_d}{d} > 1.752$$

with at most finitely many exceptions. (No exceptions of degree greater than 8.)

This is not as good as Smyth's result. It, however, follows immediately from a computation, as in Section 2, which shows that

$$\Omega[0, 1/200] < \frac{1}{201.752}$$

and gives the factors of an example which yields the above upper bound.

5. Open Problems.

There are a myriad of open problems in and around integer Chebyshev polynomials. We formulate a few of them as questions.

Q1. Find a reasonable algorithm for exactly computing integer Chebyshev polynomials on $[0, 1]$ that would work up to, say, degree 200.

Q2. Are the integer Chebyshev polynomials eventually unique?

Q3. Do the integer Chebyshev polynomials on $[0, 1]$ have all their roots in $[0, 1]$?

Q4. Determine $\Omega[0, \alpha]$ exactly for any $0 < \alpha < 4$.

Q5. Determine the limit (or the limsup, if the limit does not exist) of

$$(\Omega[0, 1/m])^{-1} - m.$$

Q6. Are all the irreducible factors of the integer Chebyshev polynomials on $[0, 1]$ forced to be factors by as in the first Lemma ? That is, are all irreducible factors q of the form

$$q(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$$

with

$$|a_k|^{1/k} < (\Omega[0, 1])^{-1}?$$

Q7. Show that there exist infinitely many irreducible polynomials with integer coefficients which divide an n-th integer Chebyshev polynomial on $[0, 1]$ for some n.

6. Prouhet-Tarry-Escott Problem.

Conjecture.

For any N there exists $p \in Z[x]$ (a polynomial with integer coefficients) so that

$$p(x) = (x-1)^N q(x) = \Sigma a_k x^k$$

and

$$S(p) := \Sigma |a_k| = 2N.$$

Almost equivalently (though not quite obviously)

$$\|p\|_{L^2\{|z|=1\}} = \sqrt{2N}.$$

The Basis for the Conjecture.

$x^{\alpha_1} + \ldots + x^{\alpha_N} - x^{\beta_1} - \ldots - x^{\beta_N} = 0((x-1)^N).$

For $N = 2, \ldots, 10$ with

$$[\alpha_1, \ldots, \alpha_N] \quad \text{and} \quad [\beta_1, \ldots, \beta_N]$$

$[0, 3] = [1, 2]$

$[1, 2, 6] = [0, 4, 5]$

$[0, 4, 7, 11] = [1, 2, 9, 10]$

$[1, 2, 10, 14, 18] = [0, 4, 8, 16, 17]$

$[0, 4, 9, 17, 22, 26] = [1, 2, 12, 14, 24, 25]$

$[0, 18, 27, 58, 64, 89, 101]$

$= [1, 13, 38, 44, 75, 84, 102]$

$[0, 4, 9, 23, 27, 41, 46, 50]$

$= [1, 2, 11, 20, 30, 39, 48, 49]$

$[0, 24, 30, 83, 86, 133, 157, 181, 197]$

$= [1, 17, 41, 65, 112, 115, 168, 174, 198]$

$[0, 3083, 3301, 11893, 23314, 24186, 35607,$

$44199, 44417, 47500] =$

$[12, 2865, 3519, 11869, 23738, 23762, 35631,$

$43981, 44635, 47488]$

• The size 10 example illustrates the problems inherent with searching for a solution.

Partial History.

- Euler
- Prouhet (1851)
- Tarry (1910) - Small Examples
- Escott (1910) - Small Examples
- Letac (1941) - Size 9 and 10
- Gloden (1946) - Size 9 and 10
- Smyth (Math Comp. 1991) - Size 10 generalized.

Diophantine Form

Find distinct integers $[\alpha_1, \ldots, \alpha_N]$ and $[\beta_1, \ldots, \beta_N]$ so that

$$
\begin{aligned}
&\alpha_1 + \ldots + \alpha_N = \beta_1 + \ldots + \beta_n \\
&\alpha_1^2 + \ldots + \alpha_N^2 = \beta_1^2 + \ldots + \beta_n^2 \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\alpha_1^{N-1} + \ldots + \alpha_N^{N-1} = \beta_1^{N-1} + \ldots + \beta_N^{N-1}
\end{aligned}
$$

More Open Questions.

- The problem is completely open for $N \geq 11$.

- We computed extensively on $N = 11$ to show no (symmetric) solutions of degree ≤ 745.

The Weak Prouhet-Tarry-Escott Problem.

Problem. For fixed N find $p \in Z[x]$

$$p(x) = (x-1)^N q(x) = \Sigma a_k x^k$$

that minimizes

$$S(p) = \Sigma |a_i|$$

or

$$S^2(p) = (\Sigma |a_i|^2)^{1/2}$$

• Solving $S(p) = |S^2(p)|^2 = 2N$ is the Prouhet-Tarry-Escott-Problem and is the big prize.

• Showing that there exist

$$\{p_N\} = \{(x-1)^N q(x)\}$$

so that

$$S(p_N) = o(N \log N)$$

is also a big prize.

• This shows that the "Easier Waring Problem" is easier than the "Waring Problem" (At the moment.)

• That is: it requires essentially fewer powers to write every integer as sums and differences of Nth powers than just as sums of Nth powers. (Fuchs and Wright, Quart. J. Math. 1936).

• It is known that

$$S((x-1)^N q(x)) \le \frac{N^2}{2}$$

is possible.

Any improvement would be a major step.

• If we demand that p has a zero of order N but not $N+1$ at 1 then

$$S(p) = 0((\log N)N^2)$$

is possible (Hua).

Any improvement would be interesting.