MÜNTZ POLYNOMIALS

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Müntz's Theorem.

A very attractive variant of Weierstrass' theorem characterizes exactly when the linear span of a system of monomials

$$\mathcal{M} := \{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0,1]$ or $L_2[0,1]$.

Müntz's Theorem in $C[0,1]$. *Suppose $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct positive real numbers not converging to* 0*. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ in the uniform norm if and only if

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

This theorem follows by a simple trick from the L_2 version of the theorem.

Müntz's Theorem in $L_2[0,1]$. *Suppose $\{\lambda_i\}_{i=0}^{\infty} \subset (-1/2, \infty)$ is a sequence of distinct real numbers not converging to $-1/2$. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_2[0,1]$ if and only if

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty.$$

Orthonormal Müntz-Legendre polynomials.

We can orthonormalize

$$\operatorname{span}\{x^{\lambda_0}, \dots, x^{\lambda_n}\}$$

Define L_n^*, the n-th orthonormal Müntz-

Legendre polynomial defined by

$$L_n(x) := \frac{1}{2\pi i}\int_\Gamma \prod_{k=0}^{n-1} \frac{t+\overline{\lambda}_k+1}{t-\lambda_k}\frac{x^t dt}{t-\lambda_n}$$

$$= \sum_{k=0}^{n} c_{k,n} x^{\lambda_k}, \qquad x \in (0,\infty)$$

with

$$c_{k,n} := \frac{\prod_{j=0}^{n-1}(\lambda_k+\overline{\lambda}_j+1)}{\prod_{j=0,j\neq k}^{n}(\lambda_k-\lambda_j)}$$

and

$$L_n^* := (1+\lambda_n+\overline{\lambda}_n)^{1/2} L_n.$$

Then we get an orthonormal system, that is,

$$\int_0^1 L_n^*(x)\overline{L_m^*(x)}dx = \delta_{m,n}, \quad m,n = 0,1,\dots .$$

Proof of Müntz's Theorem. We consider the approximation to x^m by elements of

$$\mathrm{span}\{x^{\lambda_0}, \ldots, x^{\lambda_{n-1}}\}$$

in $L_2[0,1]$

For L_n^*, the n-th orthonormal Müntz-Legendre polynomial we have

$$L_n^*(x) = \sum_{i=0}^{n-1} a_i x^{\lambda_i} + a_n x^m$$

where

$$|a_n| = \sqrt{1+2m} \prod_{i=0}^{n-1} \left| \frac{m+\lambda_i+1}{m-\lambda_i} \right|.$$

It follows from $\|L_n^*\|_{L^2[0,1]} = 1$ and orthogonality that L_n^*/a_n is the error term in the best $L_2[0,1]$ approximation to x^m from

$$\operatorname{span}\{x^{\lambda_0}, \ldots, x^{\lambda_{n-1}}\}.$$

Therefore

$$\min_{b_i \in \mathbb{C}} \left\| x^m - \sum_{i=0}^{n-1} b_i x^{\lambda_i} \right\|_{L^2[0,1]}$$

$$= \frac{1}{|a_n|} = \frac{1}{\sqrt{1+2m}} \prod_{i=0}^{n-1} \left| \frac{m-\lambda_i}{m+\lambda_i+1} \right|.$$

So, for $m \neq \lambda_i$,

$$x^m \in \overline{\operatorname{span}}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

(where $\overline{\operatorname{span}}$ denotes the $L_2[0,1]$ closure of the span) if and only if

$$\limsup_n \prod_{i=0}^{n-1} \left| 1 - \frac{2m+1}{m+\lambda_i+1} \right| = 0.$$

$\square$

Full Müntz Theorem on $[a, b]$, $a > 0$. *Let $\{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers. Then*

$$\overline{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} = C[a, b]$$

where $\overline{span}$ means the closure of the span in the uniform norm on $[a, b]$, if and only if

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty.$$

Müntz with Complex Exponents. *Suppose $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of complex numbers satisfying*

$$Re(\lambda_i) > 0, \qquad i = 1, 2, \dots .$$

If

$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = \infty$$

then

$$span\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in the set of complex-valued continuous functions on $[0,1]$.

Müntz Rationals.

A surprising and beautiful theorem, conjectured by Newman and proved by Somorjai , states that rational functions derived from any infinite Müntz system are always dense in $C[a,b]$, $a \geq 0$. More specifically we have

Denseness of Müntz Rationals. *Let* $\{\lambda_i\}_{i=0}^{\infty}$ *be any sequence of distinct real numbers. Suppose* $a \geq 0$. *Then*

$$\left\{ \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\lambda_i}} : a_i, b_i \in \mathbb{R}, \quad n \in \mathbb{N} \right\}$$

is dense in $C[a,b]$.

The proof of this theorem, primarily due to Somorjai, rests on the existence of zoomers. A function Z defined on $[a,b]$ is called an ϵ-*zoomer* ($\epsilon > 0$) at $\zeta \in (a,b)$ if

$$\begin{aligned} Z(x) &> 0, & x &\in [a,b] \\ Z(x) &\leq \epsilon, & x &< \zeta - \epsilon \\ Z(x) &\geq \epsilon^{-1}, & x &> \zeta + \epsilon \end{aligned}$$

While (approximate) δ-functions are approximate building blocks for polynomial approximations, the existence of ϵ-zoomers is all that is needed for rational approximations.

A comparison between Müntz's Theorem and this shows the power of a single division in these approximations. In what other contexts does allowing a division create a spectacularly different result.

Conjecture 1 (Newman 1978). *If $\mathcal{M}$ is any infinite Markov system on $[0,1]$ then the set of rationals*

$$\left\{\frac{p}{q} : p, q \in span\ \mathcal{M}\right\}$$

is dense in $C[0,1]$.

He calls this a "wild conjecture in search of a counterexample". It does however hold for both

$$\mathcal{M} = \{x^{\lambda_0}, x^{\lambda_1}, \dots\}, \qquad \lambda_i \geq 0$$

and

$$\mathcal{M} = \left\{\frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2} \dots\right\}$$

We give a counterexample to this conjecture. However, the characterization of the class of Markov systems for which it holds remains as an interesting question.

Newman also conjectures the non-denseness of products

Conjecture 2 (Newman 1978).

$$\{\Sigma a_i x^{i^2}\}\{\Sigma b_i x^{i^2}\}$$

is not dense in $C[0,1]$.

He speculates that this “extra” multiplication of Müntz polynomials should not carry the utility of the “extra” division.

We will show that products of two Müntz polynomials from non-dense Müntz spaces never form a dense set in $C[0,1]$.

Non-Dense Ratios of Müntz Spaces. Suppose $0 \le \lambda_0 < \lambda_1 < \cdots$. Let $a > 0$.

Show that

$$\left\{ \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{-\lambda_i}} : a_i, b_i \in \mathbb{R},\ n \in \mathbb{N} \right\}$$

is dense in $C[a, b]$, if and only if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$.

The following example is due to Boris Shekhtman and P. B.

A Markov System with Non-Dense Rationals.

We construct an infinite Markov system as follows. Consider non-negative even integers

$$0 = \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 <$$

$$\cdots < \lambda_n < \mu_n < \cdots$$

which are lacunary in the sense that there exists $q > 1$ so that

$$\frac{\mu_i}{\lambda_i} > q \quad \text{and} \quad \frac{\lambda_{i+1}}{\mu_i} > q, \qquad i = 1, 2, \dots .$$

Let $\varphi_k \in C[-1,1]$ be defined by

$$\varphi_0 := 1, \quad \varphi_{2k}(x) := x^{\mu_k}$$

and

$$\varphi_{2k+1}(x) := \begin{cases} x^{\lambda_k}, & x \geq 0 \\ -x^{\lambda_k}, & x \leq 0. \end{cases}$$

Then $\{\varphi_0, \varphi_1, \dots\}$ is a Markov system on $[-1,1]$.

But the rational functions of the form

$$\frac{\sum_{j=0}^{n} a_j \varphi_j}{\sum_{j=0}^{m} b_j \varphi_j}, \quad a_j, b_j \in \mathbb{R}, \quad n, m \in \mathbb{N}$$

are not dense in $C[-1,1]$ in the uniform norm.

Chebyshev Systems.

An essential property that polynomials of degree at most n have is that they can uniquely interpolate at $n+1$ points. This is equivalent to the fact that a polynomial of degree at most n that vanishes at $n+1$ points vanishes identically. Any $n+1$ dimensional vector space of continuous functions with this property is called a **Chebyshev space** or sometimes a Haar space.

Examples. The following are Chebyshev systems.

a] On $[0, \infty)$ with $0 = \lambda_0 < \cdots < \lambda_n$,

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}.$$

b] With $\lambda_0 < \lambda_1 < \cdots < \lambda_n$,

$$\left\{\frac{1}{x-\lambda_0}, \frac{1}{x-\lambda_1}, \ldots, \frac{1}{x-\lambda_n}\right\}.$$

c] On $(-\infty, \infty)$ with $\lambda_0 < \cdots < \lambda_n$,

$$\{e^{\lambda_0 x}, e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\}.$$

d] On $[0, \pi)$,

$$\{1, \cos\theta, \cos 2\theta, \ldots, \cos n\theta\}$$

Markov and Descartes Systems.

Chebyshev systems capture some of the essential properties of polynomials. There are two additional types of systems that capture some additional properties.

Definition (Markov System). *We say that* $\{f_0, f_1, \dots, f_n\}$ *is a Markov system on a Hausdorff space* A *if each* $f_i \in C[A]$ *and for* $m = 0, 1, \dots, n$

$$\operatorname{span}\{f_0, \dots, f_m\}$$

is a Chebyshev system. (We allow n *to tend to* $+\infty$ *in which case we call the system an infinite Markov system on* A*.)*

Definition (Descartes System). We say that $\{f_0, f_1, \dots, f_n\}$ *is a Descartes system (or order complete Chebyshev system) on* $[a, b]$ *if each* $f_i \in C[a, b]$ *and*

$$D\begin{pmatrix} g_{i_0} & g_{i_1} & \cdots & g_{i_m} \\ x_0 & x_1 & \dots & x_m \end{pmatrix} > 0$$

for any $0 \le i_0 < i_1 < \cdots < i_m$ *and* $a \le x_0 < x_1 < \cdots < x_m \le b$.

Theorem (Descartes' Rule of Signs). *Suppose $\{f_0, \dots, f_n\}$ is a Descartes system on $[a, b]$. Then the number of distinct zeros of any*

$$0 \neq f = \sum_{i=0}^{n} a_i f_i, \quad a_i \in \mathbb{R}$$

is not greater than the number of sign changes in $\{a_0, \dots, a_n\}$ (a sign change occurs exactly when $a_i a_{i+1} < 0$, $i = 0, \dots, n-1$.)

Revised Newman Conjecture. *If $\mathcal{M}$ is any infinite Descartes system on $[0, 1]$ then the set of rationals*

$$\left\{ \frac{p}{q} : p, q \in span\ \mathcal{M} \right\}$$

is dense in $C[0, 1]$.

Chebyshev Polynomials in Chebyshev Spaces.

Suppose

$$H_n := \operatorname{span}\{f_0, \dots, f_n\}$$

is a Chebyshev space on $[a, b]$ and A is a compact subset of $[a, b]$ with at least $n+1$ points. We can define the *generalized Chebyshev polynomial*

$$T_n := T_n\{f_0, \dots, f_n; A\}$$

for H_n on A by

$$T_n = c\left(f_n - \sum_{k=0}^{n-1} a_k f_k\right)$$

where the numbers $a_0, a_1, \dots, a_n \in \mathbb{R}$ are chosen to minimize

$$\left\| f_n - \sum_{k=0}^{n-1} a_k f_k \right\|_A$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that

$$\|T_n\|_A = 1$$

Theorem. *Suppose* $H_n :=$ *span*$\{f_0, \ldots, f_n\}$ *is a Chebyshev space on* $[a, b]$ *with associated Chebyshev polynomial*

$$T_n := T_n\{f_0, \ldots, f_n; [a, b]\},$$

and each f_i *is differentiable at* b. *Then*

$$\max\{|p_n'(b)| : p_n \in H_n\}$$

where $\|p_n\|_{[a,b]} \leq 1, p_n(b) = T_n(b)$ *is attained by* T_n.

Theorem (Lexicographic Property).

Suppose $\{f_0, f_1, \ldots\}$ *is a Descartes system on* $[a, b]$. *Suppose* $\lambda_0 < \lambda_1 < \ldots < \lambda_n$

and $\gamma_0 < \gamma_1 < \ldots < \gamma_n$ *are nonnegative integers satisfying*

$$\lambda_i \le \gamma_i, \qquad i = 0, 1, \ldots, n.$$

Let

$$T_n := T_n\{f_{\lambda_0}, \ldots, f_{\lambda_n}; [a, b]\}$$

and

$$S_n := S_n\{f_{\gamma_0}, \ldots, f_{\gamma_n}; [a, b]\}$$

be respectively the associated Chebyshev polynomials. Let

$$\alpha_0 < \alpha_1 < \ldots < \alpha_n \quad \text{and} \quad \beta_0 < \beta_1 < \ldots < \beta_n$$

denote the zeros of T_n *and* S_n*, respectively. Then*

$$\alpha_i \le \beta_i, \quad i = 0, \ldots, n$$

with strict inequality if $\lambda_i \ne \gamma_i$ *for at least one index* i*. (In other words the zeros of* T_n *lie to the left of the zeros of* S_n*.)*

Denseness and Zeros of Chebyshev Polynomials.

For a sequence of Chebyshev polynomials T_n associated with a fixed Markov system on $[a, b]$ we have the *mesh* of T_n is defined by

$$M_n := M_n(T_n : [a, b]) := \max_{1 \le i \le n+1} |x_i - x_{i-1}|.$$

where x_i are the zeros of T_n.

Theorem (P.B.). *If* $\mathcal{M} := \{1, f_1, f_2, \dots\}$ *is an infinite Markov system on* $[a, b]$ *with each* $f_i \in C^1[a, b]$*. Then span* $\mathcal{M}$ *is dense in* $C[a, b]$ *if and only if*

$$\lim_{n \to \infty} M_n = 0$$

where M_n *is the mesh of the associated Chebyshev polynomials.*

Corollary (Weierstrass' Theorem). *The polynomials are dense in $C[-1,1]$.*

Proof. $\mathcal{M} = \{1, x, x^2, \dots\}$ is an infinite Markov system of C^1 functions on $[-1,1]$. The associated Chebyshev polynomials are just the usual Chebyshev polynomials T_n and

$$M_n \le \frac{\pi}{n}, \qquad n = 1, 2, \dots$$

is obvious . $\square$

Denseness and Unbounded Bernstein Inequalities.

Definition (Unbounded Bernstein Inequality). *Let $\mathcal{A}$ be a subset of $C^1[a,b]$. We say that $\mathcal{A}$ has an everywhere unbounded Bernstein inequality if*

$$\sup\left\{\frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[a,b]}} : p \in \mathcal{A}, \quad p \ne 0\right\} = \infty$$

for every $[\alpha, \beta] \subset [a, b], \quad \alpha \neq \beta.$

Bernstein-Type Inequality for Chebyshev Spaces. *Let* $\{1, f_1, \ldots, f_n\}$ *be a Chebyshev system on* $[a, b]$ *such that each* f_i *is differentiable at* $x_0 \in [a, b]$*. Let*

$$T_n := T_n\{1, f_1, \ldots, f_n; [a, b]\}$$

be the associated Chebyshev polynomial. Then

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[a,b]}} \leq \frac{2}{1 - |T_n(x_0)|} |T_n'(x_0)|$$

for every $p_n \in \operatorname{span}\{1, f_1, \ldots, f_n\}$ *provided* $|T_n(x_0)| \neq 1$*.*

Characterization of Denseness by Unbounded Bernstein Inequality. *Suppose* $\mathcal{M} := \{f_0 := 1, f_1, \ldots\}$ *is an infinite Markov system on* $[a, b]$ *with each*

$f_i \in C^2[a,b]$. *Then* span $\mathcal{M}$ *is dense in* $C[a,b]$ *if and only if* span $\mathcal{M}$ *has an everywhere unbounded Bernstein inequality.*

Corollary (Weierstrass' Theorem. *The polynomials are dense in* $C[-1,1]$.