"I AM ASHAMED TO TELL YOU TO HOW MANY FIGURES I CARRIED THESE COMPUTATIONS, HAVING NO OTHER BUSINESS AT THE TIME."

Isaac Newton 1665–1666

$$\begin{aligned}\pi &= \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7}\cdots\right)\\ &= \frac{3\sqrt{3}}{4} + 24\int_0^{\frac{1}{4}} \sqrt{x-x^2}\,dx\end{aligned}$$

A ROMANTIC STORY WITH A CAST OF THOUSANDS. SET IN MANY EXOTIC LOCATIONS AND SPANNING THOUSANDS OF YEARS.

The Cast:

<u>Act 1.</u> Eudoxus, Archimedes, Tsu Chung-Chi, Leonardo of Pisa, Ludolph van Ceulen.

<u>Act 2.</u> Vieta, Wallis, Newton, Gregory, Leibnitz, Machin, Euler, Lambert, Dase, Shanks.

<u>Act 3.</u> Gauss, Legendre, Lagrange, Ramanujan.

Act 4. **Von Neumann, Salamin, Gosper, Brent, Berndt, Bailey, Kanada, Chudnovsky, Eniac, CDC, IBM, Cray, Hitachi.**

With Thanks To Co-Authors:
D. Bailey, J. Borwein,
K. Dilcher, F. Garvan.

Prehistory: $\pi = 3$

"Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about."

Old Testament, 1 Kings 7:23

$\pi \doteq 3, 3\frac{1}{7}, 3\frac{1}{8}, 3\frac{13}{81}, \ldots$

Babylonian and Egyptian.

"For a circle one meter in radius, rotation at roughly ten million revolutions per second will bring about the desired value for pi (3)"

Due to the Lorentz-Fitzgerald contraction.

(R. Norwood, 1992)

Bill No. 246, 1897. State of Indiana.

"Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side."

"In further proof of the value of the author's (E.J. Goodman, M.D.) proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his so-

lutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the American Mathematical Monthly, the leading exponent of mathematical thought in this country."

- **Passes three readings, Indiana House, 1897. (Via House Committee on Swamp Lands.)**

- **Passes first reading, Indiana Senate, 1897. (Via Senate Committee on Temperance.)**

“The case is perfectly simple. If we pass this bill which establishes a new and correct value of π, the author offers our state without cost the use of this discovery and its free publication in our school textbooks, while everyone else must pay him a royalty.”

- **Professor C.A. Waldo of Purdue intercedes and the bill is tabled.**

First Age of Pi (Archimedes)

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \qquad a_0 := 2\sqrt{3}$$

$$b_{n+1} := \sqrt{a_{n+1} \cdot b_n} \qquad b_0 := 3$$

- **This computes the length of circumscribed and inscribed regular $6 \cdot 2^n$-gons in a circle of radius one.**

Archimedes used n=4 for his estimate.

3.141$< \pi <$**3.142**	**Archimedes (287–212 BC)**
3–34 Digits	**Chinese, Arabic, etc.**
34 Digits	**Ludolph (1540–1610)**

Landmarks:

François Viète (1540–1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

(1593)

John Wallis (1616–1703)

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

(1655)

William Brouncker (1620–1684)

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cdots}}}}$$

James Gregory (1638–1675), Gottfried Wilhelm Leibnitz (1646–1716)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(1671)

Second Age of Pi (Calculus)

$\arctan x := x - x^3/3 + x^5/5 - \cdots$

$\pi = 16 \arctan(1/5) + 8 \arctan(1/239)$

100 Digits - Machin (1706)

$\pi = 20 \arctan(1/5) + 8 \arctan(3/79)$

20 Digits - Euler ($\sim$1748)

$$\pi/4 = \arctan(1/2) + \arctan(1/5) + \arctan(1/8)$$

205 Digits - Dase (1844)

Other Machin-like formulae

707(527) D.	**Shanks (1853)**
2037 D.	**ENIAC (1949)** **(von Neuman &)**
10,000 D.	**IBM 704 (1958)** **(Genuys)**
100,000 D.	**IBM 7090 (1961)** **(Shanks, Wrench)**
1-million D.	**CDC 7600 (1973)** **(Guilloud, Bouyer)**

William Shanks

Contributions to Mathematics Comprising Chiefly the Rectification of the Circle to 607 Places of Decimals. 1853

Towards the close of the year 1850 the Author first formed the design of rectifying the circle to upwards of 300 places of decimals.

He was aware at that time, that the accomplishment of his purpose would add little or nothing to his fame as a Mathematician though it might as a Computer; nor would it be productive of anything in the shape of pecuniary recompense.

17-mill.	**Symbolics 3670 (1985) Gosper**
29-mill.	**Cray-2 (1986) Bailey**
128-mill.	**HITACHI (1987) Kanada et al.**
536-mill.	**HITACHI S-810/80 (1989) Kanada**
1-billion	**IBM 3090/CRAY 2 (1989) Chudnovsky**^{2}
1-billion	**(1989) Kanada**
2-billion	**(1990) Chudnovsky**^{2}

The State of our Knowledge

We know that ...

- π **is irrational.** **Lambert (1761)**

- π **not alg.** **Lindemann (1882)**

- π **is not Liouville.** **Mahler (1953)**

- $\pi + \sqrt{2}\log 2$ **not alg.** **Baker (1966)**

- **The first few billion digits distribute regularly.**

- **123456789 appears starting at the 523,551,502th digit.**

- **The billionth digit of π is 9.**

The State of Our Ignorance

We do not know...

- **If the decimal expansion of π contains infinitely many 2's (or anything else about normality).**

- **If the continued fraction expansion is unbounded.**

- **If $\pi + e$ is transcendental (or even irrational).**

- **If $\pi/\log\pi$ is transcendental (or even irrational).**

How Big is Three Billion

- 3 billion (12pt) digits go from Halifax to Vancouver.

- 3 billion digits fill four football fields.

- 3 billion digits read off at 1 digit/second take a century.

- 3 billion is roughly the U.S. national debt in thousand dollar bills.

- 3 billion digits fill roughly 700 Bibles.

High Order Iterative Algo's

Arithmetic-geometric mean iteration: If

$$a_{n+1} := \tfrac{a_n+b_n}{2} \qquad a_0 := 1$$

and

$$b_{n+1} := \sqrt{a_n b_n} \qquad b_0 := \sqrt{2}\ \ (x).$$

Then

$$\lim a_n = \lim b_n = \frac{\pi/2}{\int_0^1 \frac{dt}{\sqrt{1-t^4}}}$$

$$= \frac{\pi/2}{\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-x^2)sin^2\theta}}}$$

In 1799, Gauss observed this purely numerically and wrote that this result

“will surely open a whole new field of analysis.”

Equivalent Modular Parameterization

$$\theta_3(q^2) = \frac{\theta_3(q) + \theta_4(q)}{2}$$

and

$$\theta_4(q^2) = \sqrt{\theta_3(q)\theta_4(q)}$$

where

$$\theta_3(q) \;=\; \sum_{-\infty}^{\infty} q^{n^2}$$

$$\theta_4(q) \;=\; \sum_{-\infty}^{\infty} (-q)^{n^2}$$

These are modular forms.

Three Algorithms for Pi.

Quartic Algorithm. Let $\alpha_0 := 6 - 4\sqrt{2}$**,** $y_0 := \sqrt{2} - 1$**,**

$$y_{n+1} := \frac{1 - \sqrt[4]{1 - y_n^4}}{1 + \sqrt[4]{1 - y_n^4}}$$

$$\alpha_{n+1} := (1 + y_{n+1})^4 \alpha_n$$

$$-2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2)$$

Then

$$\alpha_n \to \frac{1}{\pi}$$

quartically.

- **15 iterations give 2-billion digits.**

Cubic Algorithm

Let $a_0 := 1,\ b_0 := \frac{\sqrt{3}-1}{2}$. **If**

$$a_{n+1} := \frac{a_n + 2b_n}{3}$$

and

$$b_{n+1} := \sqrt[3]{\frac{b_n(a_n^2 + a_n b_n + b_n^2)}{3}}$$

then

$$\pi = \frac{3 \lim_{n\to\infty} a_n^2}{1 - \sum_{n=0}^{\infty} 3^{n+1}\left(a_n^2 - a_{n+1}^2\right)}$$

- **The convergence is cubic. Twenty two terms gives more than two billion digits.**

Quintic Algorithm

Let $s_0 := 5(\sqrt{5} - 2)$ **,** $\alpha_0 := 1/2$**.**

$$s_{n+1} := \frac{25}{(z + x/z + 1)^2 s_n}$$

$x := 5/s_n - 1, \quad , y := (x - 1)^2 + 7$**,**

$$z := \sqrt[5]{x\left(y + \sqrt{y^2 - 4x^3}\right)/2}$$

and

$$\alpha_{n+1} := s_n^2 \alpha_n$$
$$-5^n \left\{(s_n^2 - 5)/2 + \sqrt{s_n^2 - 2s_n + 5}\right\}$$

Then $1/\alpha_n$ **converges quintically to** π**.**

- **The first three iterations give 5,31 and 166 digits.**
- **Fourteen iterations give over 8-billion digits.**
- **This is based on a solvable modular equation due to Ramanujan.**

A Cubic AGM

Let $a_0 := 1$ **and** $b_0 := x$

$$a_{n+1} := \frac{a_n+2b_n}{3}$$

and

$$b_{n+1} := \sqrt[3]{\frac{b_n(a_n^2+a_nb_n+b_n^2)}{3}}$$

then the common limit is

$$F(x) = \frac{1}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x^3\right)}$$

- **The convergence is cubic.**

Proof: Opaque:

$$ {}_2F_1(1/3, 2/3; 1; x) $$

satisfies

$$ y'' + \left(\frac{1}{x} + \frac{1}{x-1}\right) y' - \frac{4}{9x(1-x)} y = 0 $$

The equivalent cubic transformation is beautiful:

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right) $$

$$ = \frac{3}{1+2x}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-x}{1+2x}\right)^3\right) $$

A Cubic AGM cont.

$$\pi = \frac{3 \lim_{n\to\infty} a_n^2}{1 - \sum_{n=0}^{\infty} 3^{n+1}\left(a_n^2 - a_{n+1}^2\right)}$$

where a_n is as above, with

$$b_0 := \frac{\sqrt{3}-1}{2}.$$

- **Taking $k+1$ terms of the sum and limit gives a cubically convergent algorithm.**

- **Twenty one terms gives three billion digits.**

Equivalent Modular Parameterization

If

$$L(q) = \sum_{-\infty}^{\infty} q^{m^2+mn+n^2}$$

and

$$M(q) = \frac{3L(q^3) - L(q)}{2}.$$

Then

$$L(q^3) = \frac{L(q) + 2M(q)}{3}$$

and

$$M(q^3) =$$

$$\sqrt[3]{\frac{M(q)\left(L^2(q) + L(q)M(q) + M^2(q)\right)}{3}}$$

Some Explanations: Let

$$ {}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!}, $$

where

$$(a)_n := a(a+1)\cdots(a+n-1).$$

Let

$$F_s(x) := {}_2F_1(\frac{1}{2} - s, \frac{1}{2} + s, 1, x^2)$$

$$G_s(x) := {}_2F_1(-\frac{1}{2} - s, \frac{1}{2} + s, 1, x^2).$$

Then

$$G_s(x)F_s(\sqrt{1-x^2}) + F_s(x)G_s(\sqrt{1-x^2})$$

$$-F_s(x)F_s(\sqrt{1-x^2}) = \frac{1}{\pi}\left(\frac{\cos(\pi s)}{2(1+2s)}\right)$$

and

$$G_s(x) = (1-x^2)F_s(x) + \frac{x(1-x^2)}{1+2s}\frac{d\,F_s(x)}{d\,x}$$

With $x := 1/\sqrt{2}$ **we get**

$$\frac{\sqrt{2}}{\cos(\pi s)} F_s(1/\sqrt{2}) F_s'(1/\sqrt{2}) = \frac{1}{\pi}$$

• **If we can compute** F_s **iteratively we can compute** F_s' **and** $1/\pi$ **by a second linked iteration.**

• There are four particularly interesting cases.

$s = 0$	**Gaussian AGM.**
$s = 1/6$	**Cubic AGM.**
$s = 1/3$	**Absolute invariant.**
$s = 1/4$	**Borchardt type.**

• Inverting $F_S(x)/F_S(\sqrt{1-x^2})$ gives modular functions.

• This makes much of this an algebraic theory.

• Finding and proving these iterations can (at least in principal) be effected entirely computationally.

Quadratic s=1/4 Iteration

Let

$$a_{n+1} := \frac{a_n+3b_n}{4}, \qquad a_0 := 1$$

and

$$b_{n+1} := \sqrt{\frac{b_n(a_n+b_n)}{2}} \qquad b_0 := x.$$

Then the common limit is

$$\frac{1}{{}_2F_1^2(1/4,3/4;1;1-x^2)}$$

Caveat Emptor: Let

$$a_{n+1} := \frac{a_n + 7b_n}{8}, \qquad a_0 := 1$$

and

$$b_{n+1} := \frac{\sqrt{a_n b_n} + 3b_n}{4} \quad b_0 := x.$$

Then the common limit $g(x)$ is not differentially algebraic.

So this is probably not a familiar named function.

“a certain impression I had of mathematicians was ... that they spent immoderate amounts of time declaring each other’s work trivial.”

(Richard Preston)

“It’s about as interesting as going to the beach and counting sand. I wouldn’t be caught dead doing that kind of work.”

(***********)

“The universe contains at most 10^{51} grains of sand.”

(Archimedes)

Ramanujan Type Series

Ramanujan's remarkable series for $1/\pi$ include

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{(4 * 99)^{4n}}.$$

- **This series adds roughly eight digits per term.**
- **Gosper in 1985 computed 17 million terms of the continued fraction for π using this.**

- **Such series exist because various modular invariants are rational (which is more-or-less equivalent to identifying those imaginary quadratic fields of class number 1).**

The Chudnovskys' series with $d = -163$ is

$$\frac{1}{12\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}.$$

$\bullet Quadratic versions come from class number two imaginary quadratic fields.$

- **The largest example has $d = -427$ and adds roughly 25 digits per term.**

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A + nB)}{C^{n+1/2}}$$

where

$$A := 212175710912\sqrt{61}$$
$$+1657145277365$$

$$B := 13773980892672\sqrt{61}$$
$$+107578229802750$$

$$C := [5280(236674 + 30303\sqrt{61})]^3.$$

These series are of the form

$$\sum_{n=0}^{\infty}(a(t)+nb(t))\frac{(6n)!}{(3n)!(n!)^3}\frac{1}{(j(t))^n}$$
$$=\frac{\sqrt{-j(t)}}{\pi}$$

where

$$\begin{aligned}
b(t) &= \sqrt{t(1728-j(t))} \\
a(t) &= \frac{b(t)}{6}\left(1-\frac{E_4(t)}{E_6(t)}\left(E_2(t)-\frac{6}{\pi\sqrt{t}}\right)\right) \\
j(t) &= \frac{1728E_4^3(t)}{E_4^3(t)-E_6^2(t)}
\end{aligned}$$

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$$

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}$$

$$E_6(q) = 1 - 540 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}$$

$$q = -e^{-\pi\sqrt{t}}.$$

- **Here t is the appropriate discriminant, j is the "absolute invariant" of Klein.**

- **E_2, E_4 and E_6 are Eisenstein series.**

- **There is an unlimited number of such series with increasingly more rapid convergence.**

- **The price is that one must deal with more complicated algebraic irrationalities.**

- **A class number p field will involve p^{th} degree algebraic integers as the constants $A = a(t)$, $B = b(t)$ and $C = c(t)$ in the series.**

- **The largest class number 3 example with $d = -907$ gives 37 or 38 digits per term.**

The largest class number four sum with $d = -1555$ is

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(n!)^3} \frac{A + nB}{C^{3n}}$$

where

$$\begin{aligned}
C &= -214772995063512240 - 96049403338648032 * 5^{1/2} \\
&- 1296 * 5^{1/2}(10985234579463550323713318473 \\
&+ 4912746253692362754607395912 * 5^{1/2})^{1/2}
\end{aligned}$$

$$\begin{aligned}
A &= 63365028312971999585426220 + 28337702140800842046825600 * 5^{1/2} \\
&+ 384 * 5^{1/2}(10891728551171178200467436212395209160385656017 \\
&+ 4870929086578810225077338534541688721351255040 * 5^{1/2})^{1/2}
\end{aligned}$$

$$\begin{aligned}
B &= 7849910453496627210289749000 + 3510586678260932028965606400 * 5^{1/2} \\
&+ 2515968 * 3110^{1/2}\ (6260208323789001636993322654444020882161 \\
&+ 2799650273060444296577206890718825190235 * 5^{1/2})^{1/2}
\end{aligned}$$

- **This gives** 50 **additional digits per term.**

Deriving the Series:

- **The absolute invariant, and so the coefficients A, B, and C satisfy polynomial equations of known degree and height.**

- **Thus the problem of determining the coefficients of each series reduces to algebra and can be entirely automated.**

- **From the expressions for $j(t)$, $a(t)$, $b(t)$ it is easy to compute their values to several hundred digits.**

• The lattice basis reduction algorithm now provides the minimal polynomials for each quantity.

• In addition, a higher precision calculation actually provides a proof of the claimed identity.

• This last step requires knowing a priori bounds on the degrees and heights.

Iterative Computations.

Quartic Algorithm.

Let $\alpha_0 := 6 - 4\sqrt{2}$, $y_0 := \sqrt{2} - 1$.
Let

$$y_{n+1} := \frac{1 - \sqrt[4]{1 - y_n^4}}{1 + \sqrt[4]{1 - y_n^4}}$$

$$\alpha_{n+1} := (1 + y_{n+1})^4 \alpha_n$$

$$-2^{2n+3} y_{n+1} (1 + y_n + y_{n+1}^2)$$

Then $\alpha_n \to 1/\pi$ **quadratically.**

- **Ten billion digits requires a couple of hundred full precision additions, multiplications, divisions, and root extractions.**

- **Multiplication is the key. Division and root extraction are Newton's method.**

- **Complexity is $\mathbf{O}(\log n M(n))$ where $M(n)$ is the bit complexity of multiplication.**

- **All algebraic numbers have complexity $\mathbf{O}(M(n))$ so the "essentialness" of the log is a very interesting question.**

• The algorithm must be started from scratch for a new run.

• Two runs of different algorithms are generally executed for error checking.

• For α algebraic $\log(\alpha)$ has the same complexity.

Series Computations

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}}$$

$$\frac{1}{\pi} = \sum_{n=0}^{N} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}} + \varepsilon_N$$

$$= \frac{a_N}{b_N} + \varepsilon_N$$

where

$b_N = 2^{12N+4}$ **is an exact power of two.**

a_N **is an integer of roughly the same size.**

- **This can be restarted if a_n and b_n are stored. (So a few extra digits is easy.)**

- **Complexity is O**$((\log n)^2 M(n))$**, but in practice it's good. (Still needs an FFT.)**

- **Error checking by congruences.**

Frauds:

Gregory's series for π, truncated at $500,000$ terms gives to thirty places

$$4 \sum_{k=1}^{500,000} \frac{(-1)^{k-1}}{2k-1}$$

$$= 3.14159\underline{0}6535897932\underline{40}4626433832\underline{6}.$$

- **Only the underlined digits are wrong.**

Theorem. For integer N divisible by 4 the following asymptotic expansion holds:

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k-1}}{2k-1} \sim \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}}$$
$$= \frac{1}{N} - \frac{1}{N^3} + \frac{5}{N^5} - \cdots$$

where the coefficients are the even Euler numbers 1, -1, 5, -61, 1385, -50521....

• Gregory's series requires more terms than there are particles in the universe to compute 100 digits of π**.**

• However, with $N = 200,000$ **and correcting using the first thousand even Euler numbers gives over** $5,000$ **digits of** π**.**

Excessive Fraud:

Sum (correct to somewhere over $42 * 10^{1000}$ digits)

$$\left(\frac{1}{10^{500}}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{10^{1000}}}\right)^2 \doteq \pi.$$

- **The sum arises from an application of Poisson summation or equivalently as a modular transformation of a theta function.**

The Billionth Digit of Pi is 9:

Can you compute just the nth digit in log space and polynomial (linear) time?

- **Probably: π^2 is possible in base 2 and base 3.**

A Final Conjecture ...

Conjecture: No one will ever know the 10^{1000}th digit of π (or the 10^{51}th).

- **Eventually we must run out of time.**