LITTLEWOOD TYPE PROBLEMS ON SUB ARCS

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Littlewood's well-known and now resolved conjecture of around 1948 concerns polynomials of the form

$$p(z) := \sum_{j=1}^{n} a_j z^{k_j},$$

where the coefficients a_j are complex numbers of modulus at least 1.

It states that such polynomials have L_1 norms on the unit circle

$$\partial D := \{z \in \mathbb{C} : |z| = 1\}$$

grow at least like

$$c \log n.$$

This was proved by Konjagin and independently by McGehee, Pigno, and Smith.

Pichorides, who contributed essentially to the proof of the Littlewood conjecture, observed that the original Littlewood conjecture (when all the coefficients are from $\{0, 1\}$) would follow from a result on the L_1 norm of such polynomials on sets $E \subset \partial D$ of measure π.

Namely if

$$\int_E \left| \sum_{j=0}^{n} z^{k_j} \right| |dz| \geq c$$

for any subset $E \subset \partial D$ of measure π with an absolute constant $c > 0$, then the original Littlewood conjecture holds.

Konjagin recently gave a lovely probabilistic proof that this hypothesis fails.

He does however conjecture the following: for any *fixed* set $E \subset \partial D$ of positive measure there exists a constant $c = c(E) > 0$ depending only on E such that

$$\int_E \left| \sum_{j=0}^{n} z^{k_j} \right| |dz| \geq c(E) \,.$$

In other words the sets $E_\epsilon \subset \partial D$ of measure π in his example where

$$\int_{E_\epsilon} \left| \sum_{j=0}^{n} z^{k_j} \right| |dz| < \epsilon$$

must vary with $\epsilon > 0$.

We show that Konjagin's conjecture holds on subarcs of the unit circle ∂D.

This relates to a variety of conjectures by Erdős, Littlewood and others from the fifties concerning p of the form

$$p(z) := \sum_{n=0}^{N} c_n z^n \qquad c_n \pm 1.$$

Erdős' Conjecture. *The supremum norm of p (as above) on the boundary of the unit disc is* $> (1+\epsilon)\sqrt{N}$.

Littlewood's Other Conjecture. *There is some p (as above) so that for all z on the boundary of the unit disc*

$$C_1 < \frac{|p(z)|}{\sqrt{N}} < C_2.$$

Here C_1 *and* C_2 *are independent of N.*

2. The Main Theorem

Let $\mathcal{S}$ denote the analytic functions f on the open unit disk D that satisfy

$$|f(z)| \leq \frac{1}{(1-|z|)}\,, \qquad z \in D\,.$$

Theorem 1. *For $f \in \mathcal{S}$ with $f(0) = 1$. If λ is any arc of the circle of length ϵ,*

$$D \int_\lambda \log_+ |f(z)| d\mu(z) + \int_\lambda \log_- |f(z)| d\mu(z) \geq C.$$

and, for any $p > 0$,

$$\int_\lambda |f(z)|^p d\mu(z) > E\epsilon \exp(-pF\epsilon^{-1}).$$

Here C, D, $E > 0$ and F are absolute constants.

Nazarov has now extended this to L_0.

Proof: Let D_ϵ be a C^2 Jordan domain contained in the unit disc D. Suppose that D_ϵ contains the origin and suppose that the boundary of D_ϵ consists of two pieces: a piece of arc, λ_1, of length ϵ on the unit circle and a curve λ_2 contained strictly within the unit disc.

Let $\omega_{D_\epsilon}(z)$ be the harmonic measure of D_ϵ with respect to the point 0.

Recall that if g is a conformal map from D to D_ϵ with $g(0) = 0$ then the harmonic measure ω_{D_ϵ} is defined on Borel sets A in the boundary of D_ϵ by

$$\omega_{D_\epsilon}(A) = \mu(g^{-1}(A))$$

where μ is linear Lebesque measure on the circle normalized to give the full circle measure 1.

For any sufficiently smooth Jordan domain (as above) the harmonic measure on λ_1 is given by a distribution in the sense that

$$d\omega_{D_\epsilon}(z) = \alpha_\epsilon(z) d\mu(z)$$

where α_ϵ is strictly positive and continuous on λ_1. (On λ_2 the same distribution is integrated against the surface measure.)

The function α_ϵ is just minus the outward normal derivative of the Greens function of D_ϵ with a pole at 0 (up to normalization.) So its strict positivity is given by the Hopf Lemma. This all says that harmonic measure in this instance behaves like arc length.

Thus we may assume that D_ϵ is chosen so that there exist positive absolute constants M_1, M_2, and M_3 so that for any $f \in \mathcal{S}$

$$\int_{\lambda_2} \log|f(z)| d\omega_{D_\epsilon}(z) \leq M_1$$

and also so that

$$\int_{\lambda_1} \log|f(z)| d\omega_{D_\epsilon}(z)$$

$$\leq M_2 \int_{\lambda_1} \log|f(z)| d\mu(z)$$

$$+M_3 \int_{\lambda_1} \log_+|f(z)| d\mu(z)$$

The first assumption above follows because $\log|f(z)| \leq |\log(1-z)|$ while the second assumption is a consequence of the intermediate value theorem (applied to the integrals of $\log_+ |f(z)|$ and $\log_- |f(z)|$ separately). We may further assume that on γ_1 the measure ω_{D_ϵ} behave uniformly like arclength in the sense that

$$0 < M_4 < \frac{\omega_{D_\epsilon}(\gamma_1)}{\epsilon} < M_5$$

and that $d\omega_{D_\epsilon}(z) = \alpha_\epsilon(z) d\mu(z)$ where

$$M_6 < \alpha_\epsilon(z) < M_7$$

and M_4, M_5, M_6 and M_7 are absolute positive constants.

We are now in a position to prove the theorem.

We first prove the theorem for a fixed D_ϵ. Since $\log|f(z)|$ is subharmonic we can find a harmonic function F that agrees with $\log|f(z)|$ on the boundary of D_ϵ. This is a harmonic majorant for $\log|f(z)|$ so

$$0 = \log|f(0)| \leq F(0).$$

Thus

$$0 \leq F(0) = \int_{\lambda_2 \cup \lambda_1} F(z) d\omega_{D_\epsilon}(z)$$

$$= \int_{\lambda_2} F(z) d\omega_{D_\epsilon}(z) + \int_{\lambda_1} F(z) d\omega_{D_\epsilon}(z)$$

$$\leq M_1 + \int_{\lambda_1} \log|f(z)| d\omega_{D_\epsilon}(z)$$

$$\leq M_2 \int_{\lambda_1} \log|f(z)| d\mu(z) + M_3 \int_{\lambda_1} \log_+ |f(z)| d\mu(z)$$

$$+ M_1.$$

Here the last inequality follows from the assumptions on the contours. For any $p > 0$ we have by Jensen's inequality

$$\int_{\lambda_1} \log|f(z)| d\omega_{D_\epsilon}(z) \leq$$

$$(1/p)\omega_{D_\epsilon}(\gamma_1) \log\Big[(\omega_{D_\epsilon}(\gamma_1)^{-1} \int_{\gamma_1} |f(z)|^p d\omega_{D_\epsilon}(z)\Big].$$

Since $d\omega_{D_\epsilon}(z) = \alpha_\epsilon(z) d\mu(z)$ with α_ϵ strictly positive and continuous on λ_1 and since $|f(z)|$ is positive

$$\int_{\gamma_1} |f(z)|^p d\mu(z) > E\epsilon \exp(-Fp\epsilon^{-1}).$$

Here $E := M_6 M_4 > 0$ and $F := M_1/M_7$ are independent of ϵ and p. This completes the result for a fixed ϵ.

Up to this point M_1 and M_2, in principal, depend on ϵ. To see that we can make the estimate independent of ϵ we argue as follows. First we observe that it is sufficient to prove that the estimate is uniform for a nested sequence of arcs, λ_{ϵ_i}, whose lengths, ϵ_i, tend to zero. Here we are denoting by λ_{ϵ_i} the piece of the boundary of the domain of D_{ϵ_i} that is on the unit circle. Now suppose we choose, as we may, the domains D_{ϵ_i} so that they satisfy the conditions previously outlined and they also tend very smoothly to a circle contained in the unit disc that contains zero and is tangent to the circle at a single point. It is now an easy compactness argument to see that uniformity. This follows mostly from the fact that the normal derivatives of the Greens functions stay uniformly bounded away from zero. $\square$

It is possible to find a polynomial in the class $\mathcal{S}$ with constant coefficient 1 that is small on a subset of the unit circle of measure as close to full measure as one wishes. This method was suggested by Nazarov.

Lemma 1. *For every $r \in (0, 1/2)$ there exists a trigonometric polynomial*

$$p(z) = \sum_{j=-n}^{n} c_j z^j$$

such that $c_0 = 1$, $|c_j| < r$ and $|p(z)| < r$ everywhere on the unit circle except in a set of linear measure at most r.

Proof. The finite Riesz product

$$p(z) = \prod_{j=1}^{N} (1 + rz^{m_j} + rz^{-m_j})$$

with $m_j := 4^j$ and sufficiently large N is such an example. For $r \in (0, 1/2)$ the Riesz products tend to 0 almost everywhere on the unit circle as $N \to \infty$. $\square$

The transfinite diameter of any closed proper subset of the unit circle is less than one so

Lemma 2. *For every $R > 0$ there exists a polynomial*

$$f(z) = \sum_{k=0}^{M} a_k z^k$$

with integer coefficients and $|f(z)| < R$ everywhere on the boundary of the unit circle except possibly on a set of linear measure at most R.

Theorem 2. *For every $R > 0$ there exists a polynomial in the class $\mathcal{S}$ with constant coefficient 1*

$$f(z) = \sum_{k=0}^{M} a_k z^k$$

such that $|f(z)| < R$ everywhere on the boundary of the unit circle except possibly on a set of linear measure at most R.

Proof. Take f as is in Lemma 2 and consider $f(z^M)$ for large, as yet unspecified, M. (This does not effect the measure of the subset of $\{|z| = 1\}$ where $f(z^M)$ is small).

Now Lemma 1 can be used to systematically replace any fixed coefficients of

$f(z^M)$ of size greater than one by a coefficients of size one smaller and some additional terms with coefficients of small size.

This can be done so as to have as small an effect on the size of the exceptional set as one desires. (The required sizes of the M's depends only on the maximum size of coefficient of f and on the choice of r in Lemma 2.) $\square$