PRECISE ESTIMATES ON THE MULTIPLICITY OF ROOTS OF CERTAIN POLYNOMIALS.

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We consider the problem of minimizing the uniform norm on $[0,1]$ over polynomials p

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_j| \le 1 \,, \quad a_j \in \mathbb{C}$$

with fixed $|a_m| \neq 0$.

This is equivalent to the question of how many zeros such a polynomial can have at 1.

Particular cases include:

Polynomials with coefficients in the set $\{-1, 0, 1\}$.

Polynomials with coefficients in the set $\{0, 1\}$ on the interval $[-1, 0]$.

$$\mathcal{P}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R} \right\}$$

$$\mathcal{Z}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z} \right\}$$

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 0, 1\} \right\}$$

$$\mathcal{A}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{0, 1\} \right\}$$

So obviously

$$\mathcal{A}_n \subset \mathcal{F}_n \subset \mathcal{Z}_n \subset \mathcal{P}_n.$$

2. Number of Zeros at 1

Theorem 2.1. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1, \quad a_j \in \mathbb{C}$$

has at most

$$c\,(n(1 - \log|a_0|))^{1/2}$$

zeros at 1.

Applying Theorem 2.1 with $q(x) := x^{-n}p(x^{-1})$ gives the following:

Theorem 2.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1, \quad a_j \in \mathbb{C}$$

has at most

$$c\,(n(1 - \log|a_n|))^{1/2}$$

zeros at 1*.*

Sharpness of the above theorems, up to constants, is shown by the next result.

Theorem 2.3. *It* $\exp(-3n) \le |a_0| \le 1$, *then there exists a polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_j| \le 1 \,, \quad a_j \in \mathbb{C}$$

such that p *has a zero at* 1 *with multiplicity at least*

$$\frac{1}{5}(n\,(1 - \log|a_0|))^{1/2} - 1 \,.$$

The next two theorems treat the case $a_0 = 1$. The proofs are attractive and we will work through them. (As time allows.)

Theorem 2.4. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_n| = 1\,, \qquad |a_j| \leq 1$$

has at most $5\sqrt{n}$ zeros at 1.

Theorem 2.5. *For every $n \in \mathbb{N}$, there exists*

$$p_n(x) = \sum_{j=0}^{2n^2} a_j x^j$$

such that $a_{2n^2} = 1$; $a_0, a_1, \ldots, a_{2n^2-1}$ are real numbers of modulus less than 1; and p_n has a zero at 1 with multiplicity at least n.

Theorem 2.5 immediately implies

Corollary 2.6. *For every $n \in \mathbb{N}$, there exists a polynomial*

$$p_n(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad a_n = 1 \,,$$

$a_0, a_1, \ldots, a_n$ are real numbers of modulus less than 1, and p_n has a zero at 1 with multiplicity at least $\lfloor \sqrt{n/2} \rfloor$.

The next related result is well known:

Theorem 2.7. *There is an absolute constant* $c > 0$ *so that for every* $n \in \mathbb{N}$ *there is a* $p \in \mathcal{F}_n$ *having at least* $c\sqrt{n/\log(n+1)}$ *zeros at* 1*.*

Theorems 2.4 and 2.7 show that the right upper bound for the number of zeros a polynomial $p \in \mathcal{F}_n$ can have at 1 is somewhere between $c_1\sqrt{n/\log(n+1)}$ and $c_2\sqrt{n}$ with absolute constants $c_1 > 0$ and $c_2 > 0$.

This gap looks quite hard to close.

Our final result in this section is a simple observation about the maximal number of zeros a polynomial $p \in \mathcal{A}_n$ can have.

Theorem 2.8. *There is an absolute constant* $c > 0$ *such that every* $p \in \mathcal{A}_n$ *has at most* $c \log n$ *zeros at* -1.

Remark to Theorem 2.8. Let R_n be defined by

$$R_n(x) := \prod_{i=1}^{n} (1 + x^{a_i}) \,,$$

where $a_1 := 1$ and a_{i+1} is the smallest odd integer that is greater than $\sum_{k=1}^{i} a_k$.

It is tempting to speculate that R_n is the lowest degree polynomial with coefficients $\{0, 1\}$ and a zero of order n at -1.

This is true for $n := 1, 2, 3, 4, 5$ but fails for $n := 6$ and hence for all larger n.

3. Restricted Chebyshev Problem

Theorem 3.1. *There are absolute constants so that*

$$\exp\left(-c_1 n(1-\log|a_m|))^{1/2}\right) \le \inf_p \|p\|_{[0,1]} \le \exp\left(-c_2 n(1-\log|a_m|))^{1/2}\right) ,$$

where the inf is taken over $0 \neq p$ *of the form*

$$p(x) = \sum_{j=m}^{n} a_j x^j , \qquad |a_j| \le 1 , \quad a_j \in \mathbb{C}$$

with $|a_m| \ge \exp\left(\frac{1}{2}(1-n)\right)$.

This specializes to

Theorem 3.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \le \inf_p \|p\|_{[0,1]} \le \exp\left(-c_2\sqrt{n}\right) \,,$$

for polynomials of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_j| \le 1 \,, \quad a_n = 1 \,.$$

For the class $\mathcal{F}_n$ we have

Theorem 3.3. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right)$$

$$\leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]}$$

$$\leq \exp\left(-c_2\sqrt{n}(\log(n+1))^{-1/2}\right).$$

The approximation rate in Theorems 3.2 and 3.3 should be compared with

$$\min_{p(x):=x^n+\cdots\in\mathcal{P}_n} \|p\|_{[0,1]}^{1/n} = \frac{2^{1/n}}{4}\,,$$

and also with

$$\frac{1}{2.376\ldots} < \min_{0\neq p\in\mathcal{Z}_n} \|p\|_{[0,1]}^{1/n} < \frac{1+o(1)}{2.3605}\,.$$

The first equality above is attained by the normalized Chebyshev polynomial shifted linearly to $[0,1]$ and is proved by a simple perturbation argument. The second inequality is much harder (the exact result is open).

It is an interesting fact that the polynomials $0 \neq p \in \mathcal{Z}_n$ with the smallest uniform norm on $[0, 1]$ are very different from the usual Chebyshev polynomial of degree n.

For example, they have at least 52% of their zeros at either 0 or 1. Relaxation techniques do not allow for their approximate computation.

Likewise, polynomials $0 \neq p \in \mathcal{F}_n$ with small uniform norm on $[0, 1]$ are again quite different from polynomials $0 \neq p \in \mathcal{Z}_n$ with small uniform norm on $[0, 1]$.

The story is roughly as follows. Polynomials $0 \neq p \in \mathcal{P}_n$ with leading coefficient 1 and with smallest possible uniform norm on $[0, 1]$ are characterized by equioscillation and are given by the Chebyshev polynomials explicitly.

In contrast, finding polynomials from $\mathcal{Z}_n$ with small uniform norm on $[0, 1]$ is closely related to finding irreducible polynomials with all their roots in $[0, 1]$.

As we shall see the construction of small norm polynomials from $\mathcal{F}_n$ is governed by how many zeros such a polynomial can have at 1.

It is interesting to note that the polynomials $0 \neq p \in \mathcal{P}_n$ with leading coefficient 1 and with smallest uniform norm on $[0, 1]$ have coefficients that alternate in sign.

This also appears to be true for the analogous polynomials from $\mathcal{Z}_n$ (though this is only conjectural and probably quite hard to prove).

This is quite different from the story for $\mathcal{F}_n$. We show that for polynomials $p(-x)$ with $0 \neq p \in \mathcal{A}_n$ we get a very much larger smallest possible uniform norm on $[0,1]$.

Theorem 3.4. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\begin{aligned}
&\exp\left(-c_1 \log^2(n+1)\right) \\
&\leq \inf_{0 \neq p \in \mathcal{A}_n} \|p(-x)\|_{[0,1]} \\
&\leq \exp\left(-c_2 \log^2(n+1)\right)
\end{aligned}$$

4. Tools

In the general case the tools are:

Denote by $\mathcal{S}$ the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \le \frac{1}{1-|z|}\,, \qquad z \in D\,.$$

Theorem 4.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$|f(0)|^{c_1/a} \le \exp\left(\frac{c_2}{a}\right) \|f\|_{[1-a,1]}$$

for every $f \in \mathcal{S}$ and $a \in (0,1]$.

Hadamard Three Circles Theorem. *Suppose f is regular. Let $M(r) := \max_{|z|=r} |f(z)|$. Then for $r_1 < r < r_2$*

$$M(r)^{\log(r_2/r_1)} \le M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}.$$

Halász Lemma. *For every $k \in \mathbb{N}$, there exists a polynomial $h \in \mathcal{P}_k^c$ such that*

$$h(0) = 1\,, \quad h(1) = 0\,, \quad |h(z)| < \exp\left(\frac{2}{k}\right)$$

for $|z| \le 1$.

5. Proofs of the Main Results

Theorem 2.4. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_n| = 1\,, \qquad |a_j| \le 1$$

has at most $5\sqrt{n}$ zeros at 1.

Proof of Theorem 2.4. If p has a zero at 1 of multiplicity m, then for every polynomial f of degree less than m, we have

$$(*) \qquad a_0 f(0) + a_1 f(1) + \cdots + a_n f(n) = 0\,.$$

We construct a polynomial f of degree at most $5\sqrt{n}$, for which

$$f(n) > |f(0)| + |f(1)| + \cdots + |f(n-1)|\,.$$

Equality (*) cannot hold with this f, so the multiplicity of the zero of p at 1 is at most the degree of f.

Let T_ν be the ν-th Chebyshev poly. Let

$$g := T_0 + T_1 + \cdots + T_k \in \mathcal{P}_k\,.$$

Note that $g(1) = k+1$ and

$$g(\cos y) = 1 + \cos y + \cos 2y + \cdots + \cos ky$$
$$= \frac{\sin(k+\frac{1}{2})y + \sin\frac{1}{2}y}{2\sin\frac{1}{2}y}\,.$$

Hence, for $-1 \leq x < 1$,

$$|g(x)| \leq \frac{\sqrt{2}}{\sqrt{1-x}}\,.$$

Let $f(x) := g^4(\frac{2x}{n} - 1)$. Then $f(n) = (k+1)^4$ and

$$|f(0)|+|f(1)|+\cdots+|f(n-1)| \leq \sum_{j=1}^{n} \frac{4}{\left(\frac{2j}{n}\right)^2} < \frac{\pi^2}{6} n^2\,.$$

If $k := \lfloor (\pi^2/6)^{1/4} \sqrt{n} \rfloor$ then

$$f(n) > |f(0)| + |f(1)| + \cdots + |f(n-1)|\,.$$

In this case the degree of f is $4k \leq 5\sqrt{n}$. $\square$

Theorem 2.5. *For every $n \in \mathbb{N}$, there exists*

$$p_n(x) = \sum_{j=0}^{2n^2} a_j x^j$$

such that $a_{2n^2} = 1$; $a_0, a_1, \ldots, a_{2n^2-1}$ are real numbers of modulus less than 1; and p_n has a zero at 1 with multiplicity at least n.

Proof of Theorem 2.5. Define

$$L_n(x) := \frac{(n!)^2}{2\pi i} \int_\Gamma \frac{x^t \, dt}{\prod_{k=0}^{n} (t - k^2)}$$

where the simple closed contour Γ surrounds the zeros of the denominator in the integrand.

Then L_n is a polynomial of degree n^2 with a zero of order n at 1.

Also, by the residue theorem,

$$L_n(x) = 1 + \sum_{k=1}^{n} c_{k,n} x^{k^2}$$

where

$$c_{k,n} = \frac{(n!)^2}{\prod_{j=0, j\neq k}^{n} (k^2 - j^2)} = \frac{(-1)^k 2(n!)^2}{(n-k)!(n+k)!}$$

It follows that

$$c_{k,n} \le 2\,, \qquad k = 1, 2, \dots, n$$

Hence,

$$q_n(x) := \frac{L_n(x) + L_n(x^2)}{2}$$

is a polynomial of degree $2n^2$ with real coefficients and with a zero of order n at 1. Also q_n has constant coefficient 1 and each of its remaining coefficients is a real number of modulus less than 1. Now let $p_n(x) := 2x^{n^2} q_n(1/x)$. $\square$

Proof of Theorem 2.8. Suppose $P \in \mathcal{A}_n$ has m zeros at -1. Then $(1+x)^m$ divides P. On evaluating the above at 1 we see that $n \geq 2^m - 1$ and the result follows. $\square$

6. Comments

There is an obvious interval dependence in the problem of finding minimal elements from $\mathcal{F}_n$.

On any interval $[0, \delta]$ with $\delta < 1/2$ the only polynomials from $\mathcal{F}_n$ with minimal uniform norm are $\pm x^n$.

On $[0, 1/2]$ all of $\pm x^n$ and $\pm(x^n - x^{n-1})$ are extremals.

On any interval $[0, \delta]$ with $\delta > 1/2$ the polynomials $\pm(x^n - x^{n-1})$ work better than x^n, so the nature of the extremals change at $1/2$.

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