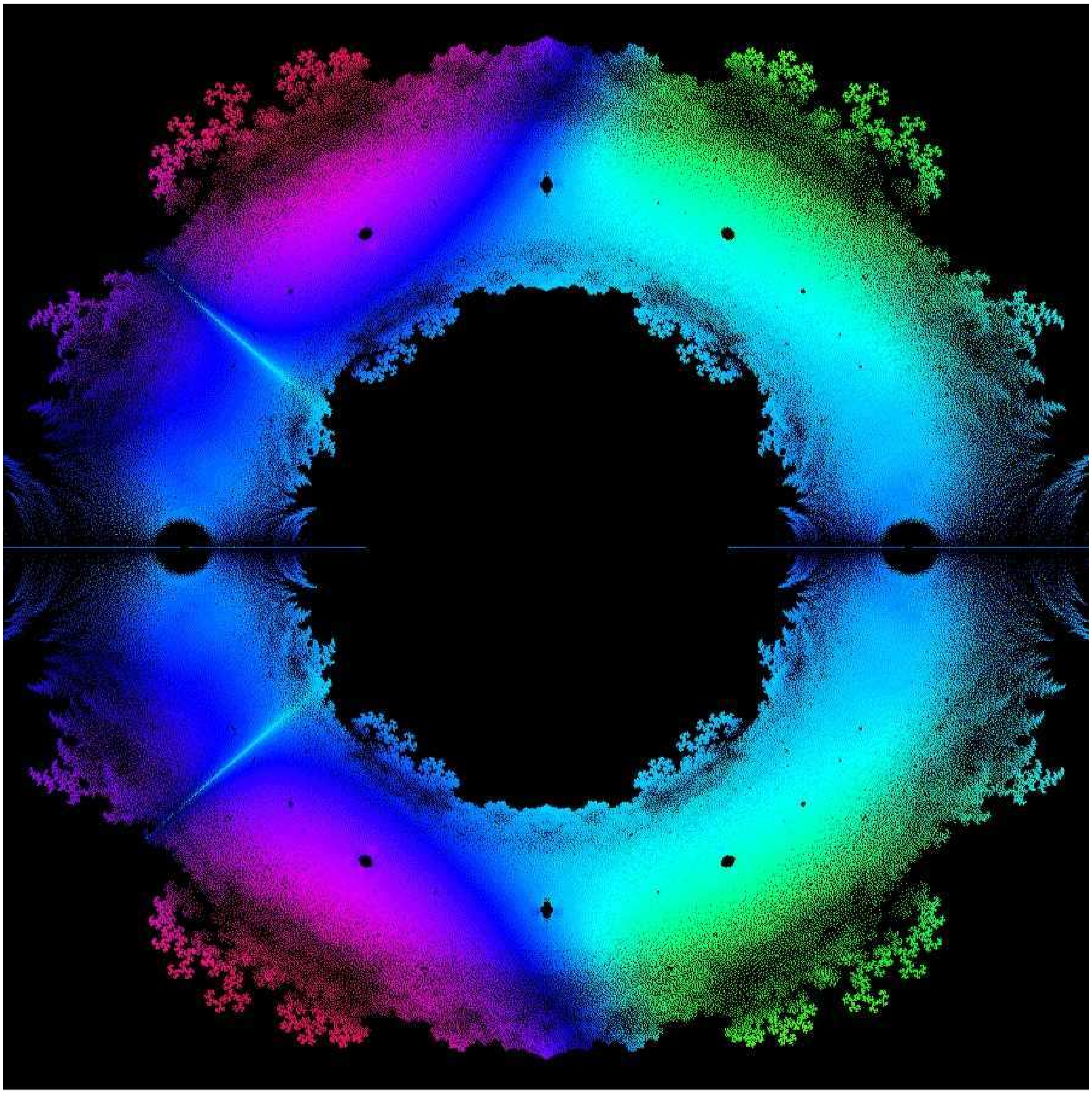
The Million Dollar Millennium Prize Problems

Peter Borwein


RESEARCH INSTITUTE
UNANSWERED QUESTIONS
UNQUESTIONED ANSWERS
S. Harris

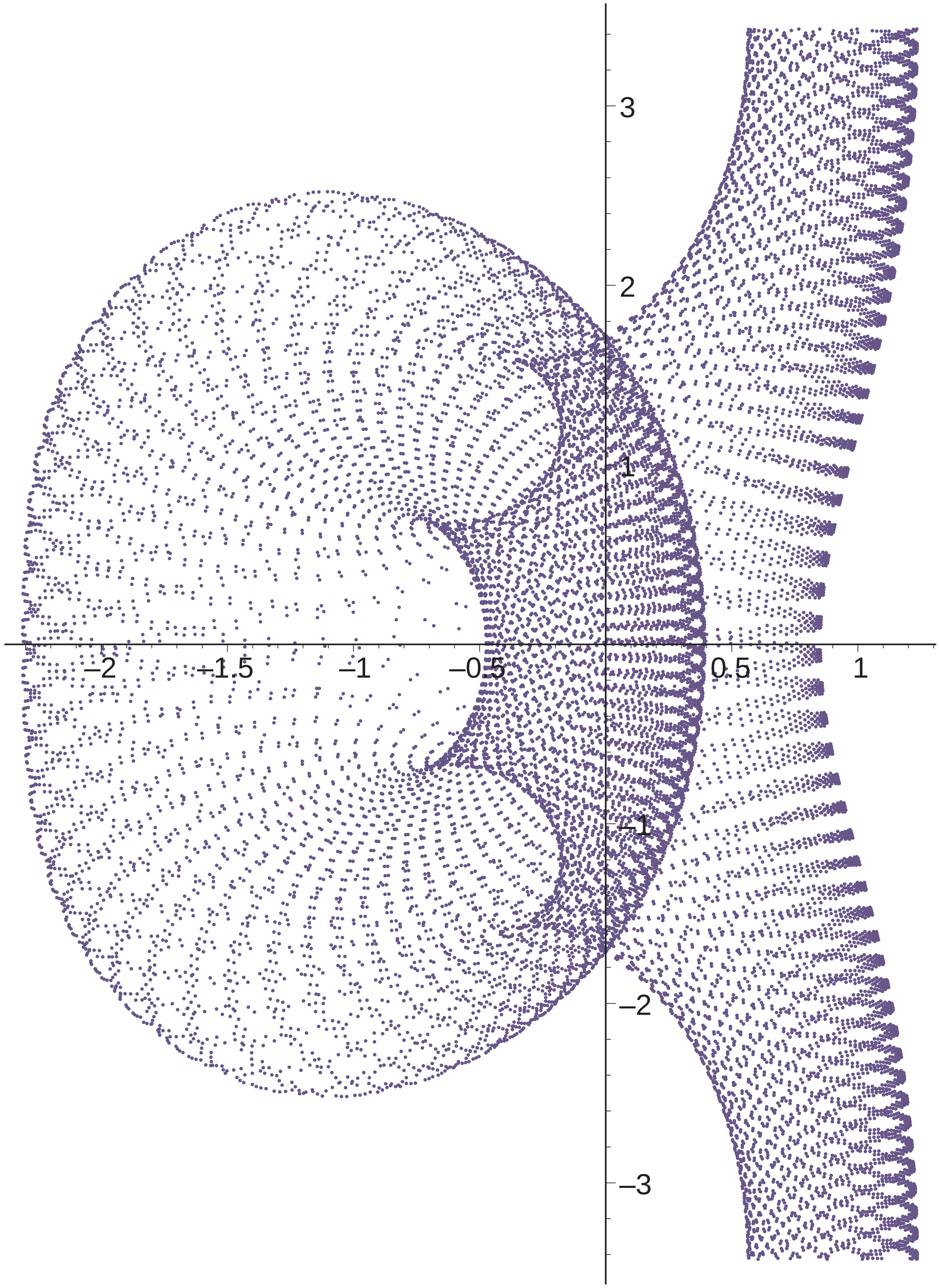


" Curiosity is part of human nature. Unfortunately, the established religions no longer provide the answers that are satisfactory, and that translates into a need for certainty and truth.

And that is what makes mathematics work, makes people commit their lives to it.

It is the desire for truth and the response to the beauty and elegance of mathematics that drives mathematicians"

Landon Clay (Wealthy mutual fund magnate and Harvard English graduate.)



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Millennium Prize Problems

"To celebrate mathematics in the new millennium, CMI identifies seven old and important mathematics questions that resisted all past attempts to solve them. Clay Mathematics Institute designates the $7 million prize fund for their solution, with $1 million allocated to each Millennium Prize Problem.

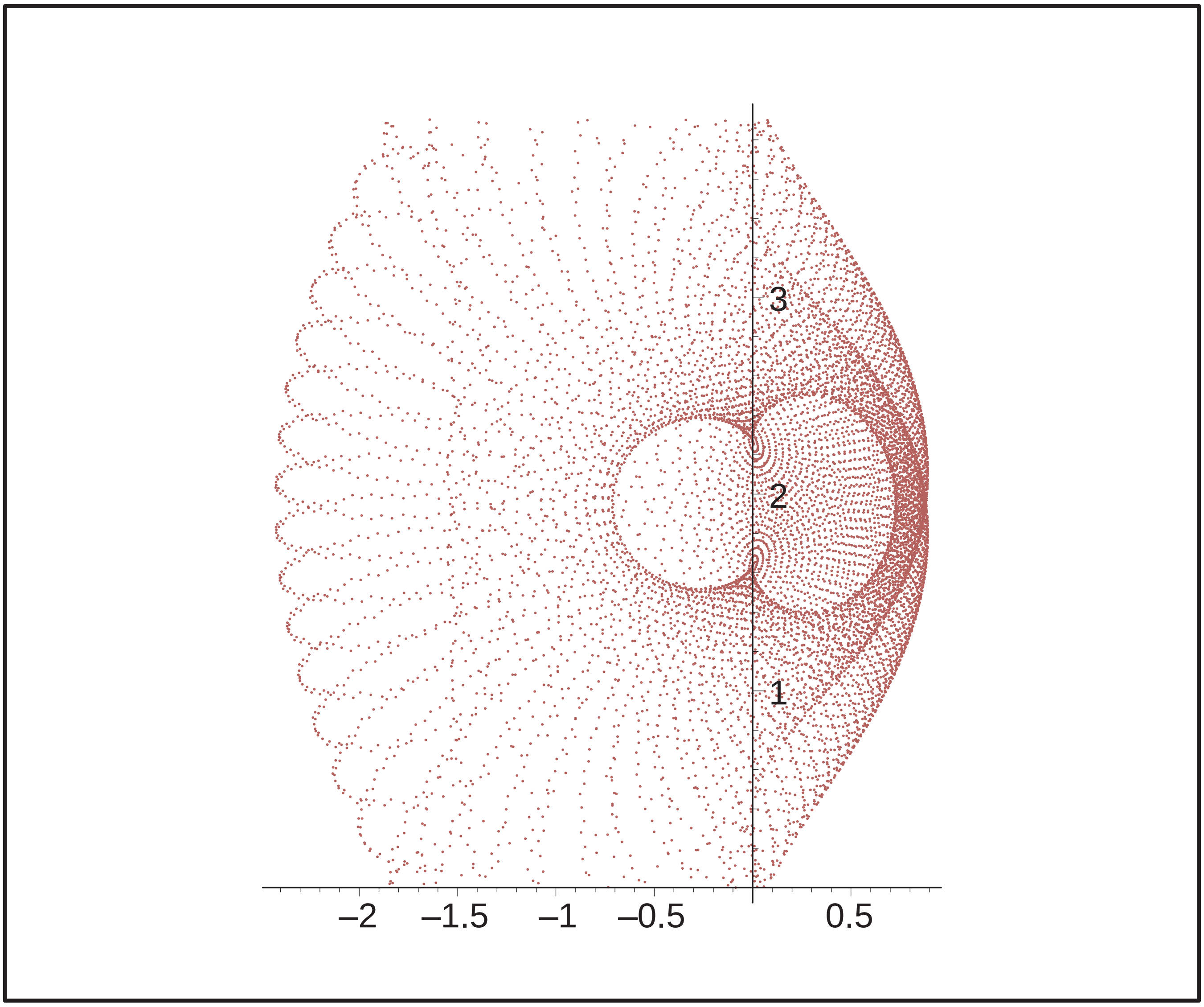
The Clay Mathematics Institute (CMI) is a private, non-profit foundation, dedicated to increase and to disseminate mathematical knowledge. The formation of CMI grew from the vision of Boston businessman Landon T. Clay working together with mathematician Arthur M. Jaffe: mathematics embodies the quintessence of human knowledge; mathematics reaches into every field of human endeavor; and the frontiers of mathematical understanding evolve today in deep and unfathomable ways.

Fundamental advances in mathematical knowledge go hand in hand with discoveries in all fields of science.

Technological applications of mathematics underpin our daily life, including our ability to communicate and to travel, our health and well-being, our security, and our global prosperity.

The evolution of mathematics today will remain a central ingredient in shaping our world tomorrow. To appreciate the scope of mathematical truth challenges the capabilities of the human mind.

CMI attempts to further the beauty, the power, and the universality of mathematical thought. Toward this end, CMI currently pursues a series of programs."



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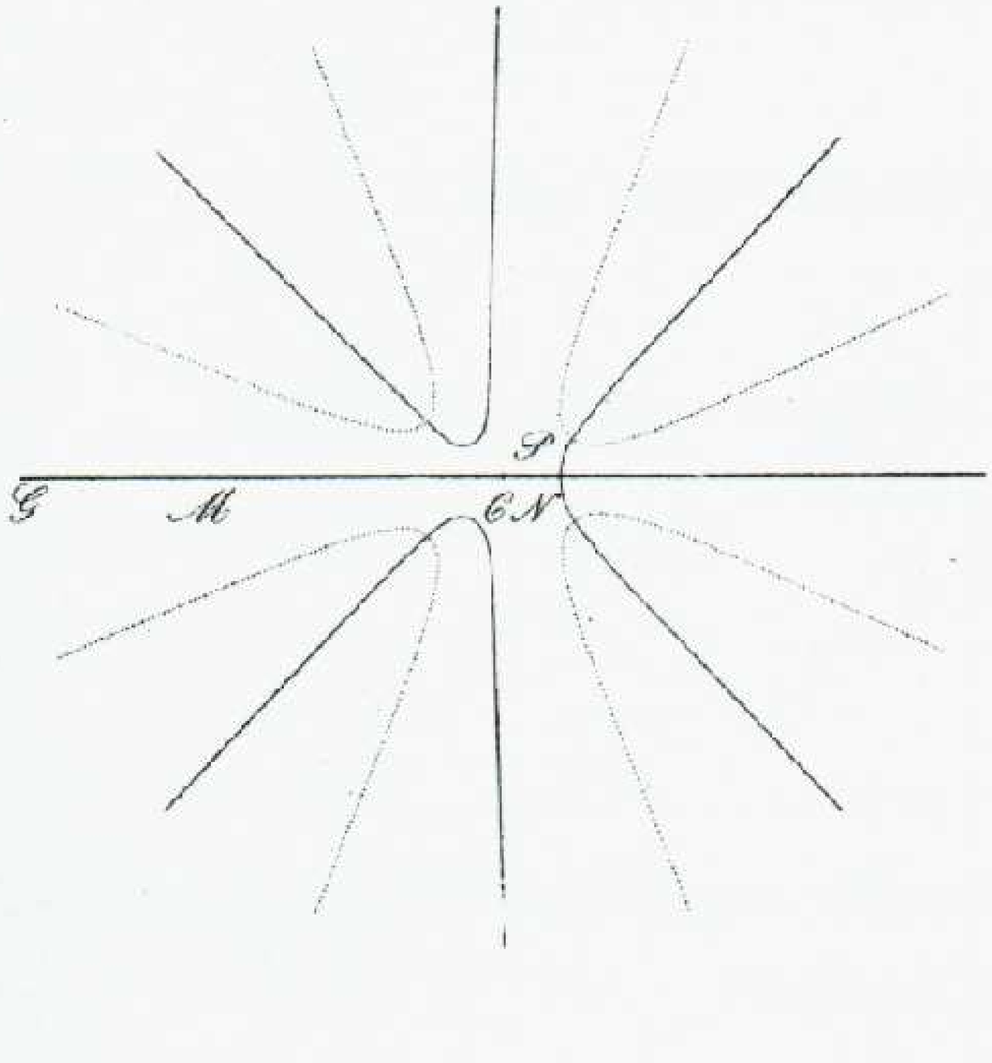
"It still remains true that, with negative theorems such as this, transforming personal convictions into objective ones requires deterringly detailed work. To visualize the whole variety of cases, one would have to display a large number of equations by curves; each curve would have to be drawn by its points, and determining a single point alone requires lengthy computations. You do not see from Fig. 4 in my first paper of 1799 , how much work was required for a proper drawing of that curve."

K. F. Gauss (1777-1855)

Fig. 4.



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> Sum(1/n^6+1/n^8,n=1..infinity);
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$$\sum_{n=1}^{\infty}\left(\frac{1}{n^6}+\frac{1}{n^8}\right)$$

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> sum(1/n^6+1/n^8,n=1..infinity);
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$$\frac{1}{945}\pi^6+\frac{1}{9450}\pi^8$$

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> Int(exp(x)*sin(x),x);
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$$\int e^x \sin(x)\,dx$$

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> int(exp(x)*sin(x),x);
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$$-\frac{1}{2}e^x\cos(x)+\frac{1}{2}e^x\sin(x)$$

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>
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The Problems

1. BIRCH AND SWINNERTON-DYER CONJECTURE

Mathematicians have always been fascinated by the problem of describing all solutions in whole numbers x,y,z to algebraic equations like

$$x^2 + y^2 = z^2$$

Euclid gave the complete solution for that equation, but for more complicated equations this becomes extremely difficult.

Indeed, in 1970 Yu. V. Matiyasevich showed that Hilbert's tenth problem is unsolvable, i.e., there is no general method for determining when such equations have a solution in whole numbers.

But in special cases one can hope to say something. When the solutions are the points of an abelian variety, the Birch and Swinnerton-Dyer conjecture asserts that the size of the group of rational points is related to the behavior of an associated zeta function $\zeta(s)$ near the point s=1.

In particular this amazing conjecture asserts that if $\zeta(1)$ is equal to 0, then there are an infinite number of rational points (solutions), and conversely, if $\zeta(1)$ is not equal to 0, then there is only a finite number of such points.

Too Obscure to EVEN STATE

2. HODGE CONJECTURE

In the twentieth century mathematicians discovered powerful ways to investigate the shapes of complicated objects. The basic idea is to ask to what extent we can approximate the shape of a given object by gluing together simple geometric building blocks of increasing dimension.a

This technique turned out to be so useful that it got generalized in many different ways, eventually leading to powerful tools that enabled mathematicians to make great progress in cataloging the variety of objects they encountered in their investigations.

Unfortunately, the geometric origins of the procedure became obscured in this generalization.

In some sense it was necessary to add pieces that did not have any geometric interpretation.

The Hodge conjecture asserts that for particularly nice types of spaces called projective algebraic varieties, the pieces called Hodge cycles are actually (rational linear) combinations of geometric pieces called algebraic cycles.

Weather Forecasting!

3. NAVIER-STOKES EQUATIONS

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet.

Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations.

Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Euler's equations of motion

We are now in a position to apply the principle of linear momentum to a small 'dyed' blob of fluid of volume δV. Allowing for the presence of a gravitational body force per unit mass $\boldsymbol{g}$, the total force on the blob is

$$(-\nabla p + \rho \boldsymbol{g})\, \delta V.$$

This force must be equal to the product of the blob's mass (which is conserved) and its acceleration, i.e. to

$$\rho\, \delta V \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}.$$

We thus obtain

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}, \tag{1.12}$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

as the basic equations of motion for an ideal fluid. They are known as *Euler's equations*, and written out in full they become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

2.2. The equations of viscous flow

So far we have considered the motion of fluids of small viscosity. Yet there is more to the subject than this, including the opposite extreme of very viscous flow (Chapter 7). It is time, then, to take a more balanced—if brief—look at viscous flow as a whole.

The Navier–Stokes equations

Suppose that we have an incompressible Newtonian fluid of constant density ρ and constant viscosity μ. Its motion is governed by the *Navier–Stokes equations*†

$$\begin{gathered}\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{g}, \\ \nabla \cdot \boldsymbol{u} = 0.\end{gathered} \tag{2.3}$$

These differ from the Euler equations (1.12) by virtue of the viscous term $\nu\nabla^2\boldsymbol{u}$, where ∇^2 denotes the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

The no-slip condition

Observations of real (i.e. viscous) fluid flow reveal that both normal and tangential components of fluid velocity at a rigid boundary must be equal to those of the boundary itself. Thus if the boundary is at rest, $\boldsymbol{u} = 0$ there. The condition on the tangential component of velocity is known as the *no-slip condition,* and it holds for a fluid of any viscosity $\nu \neq 0$, no matter how small ν may be.

"I hate getting all these Canadian coins, but I guess that's the price of living in Toronto."

A Canadian Connection

Formulated in large measure by Stephen Cook (1971).

4. P VS NP

Suppose that you are organizing housing accommodations for a group of four hundred university students. Space is limited and only one hundred of the students will receive places in the dormitory. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice.

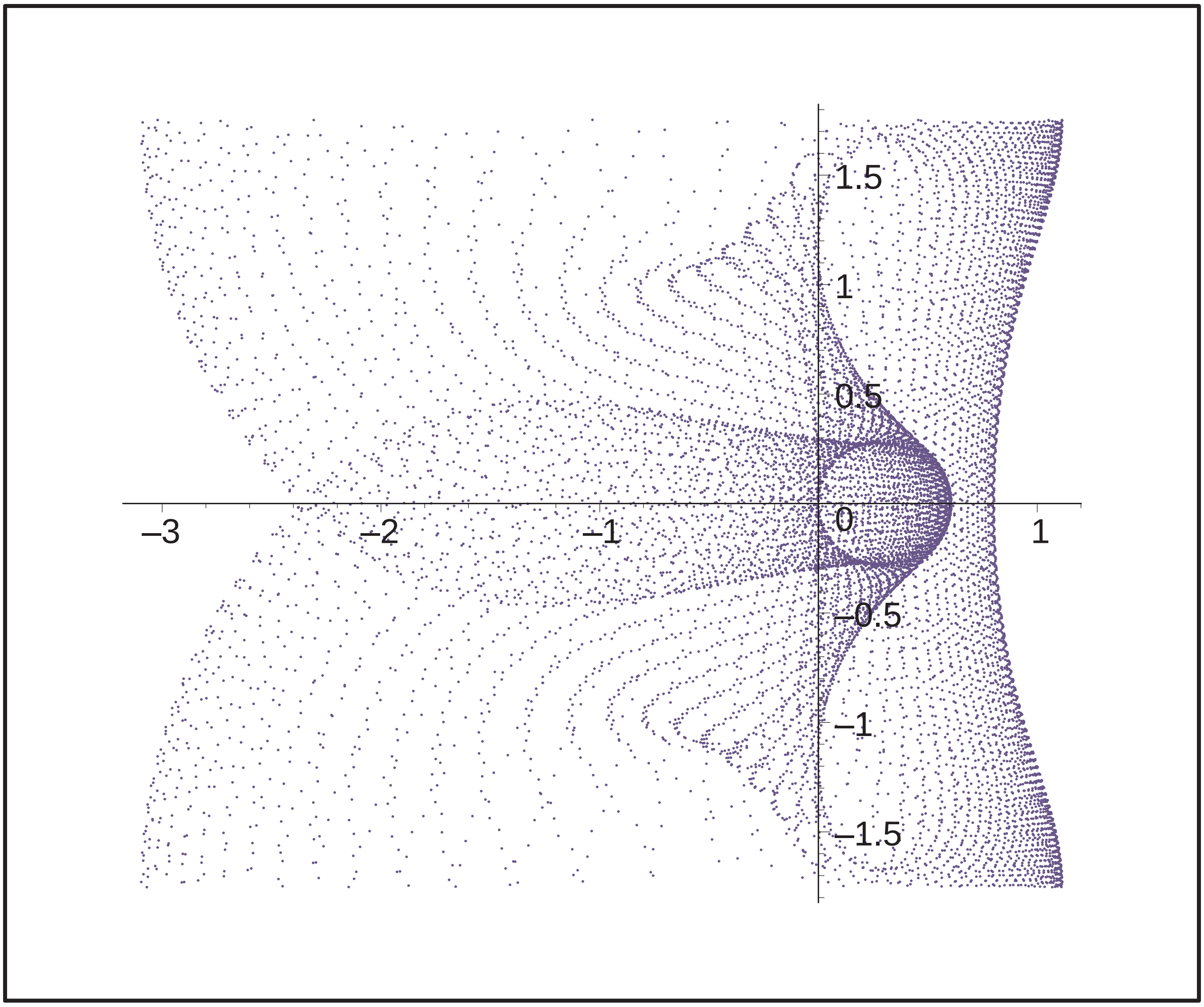
This is an example of what computer scientists call an NP-problem, since it is easy to check if a given choice of one hundred students proposed by a coworker is satisfactory (i.e., no pair from taken from your coworker's list also appears on the list from the Dean's office), however the task of generating such a list from scratch seems to be so hard as to be completely impractical.

Indeed, the total number of ways of choosing one hundred students from the four hundred applicants is greater than the number of atoms in the known universe!

Thus no future civilization could ever hope to build a supercomputer capable of solving the problem by brute force; that is, by checking every possible combination of 100 students. However, this apparent difficulty may only reflect the lack of ingenuity of your programmer.

In fact, one of the outstanding problems in computer science is determining whether questions exist whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure.

Problems like the one listed above certainly seem to be of this kind, but so far no one has managed to prove that any of them really are so hard as they appear, i.e., that there really is no feasible way to generate an answer with the help of a computer. Stephen Cook and Leonid Levin formulated the P (i.e., easy to find) versus NP (i.e., easy to check) problem independently in 1971.



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The First to Go?

5. POINCARE CONJECTURE

If we stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface.

On the other hand, if we imagine that the same rubber band has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut.

We say the surface of the apple is "simply connected," but that the surface of the doughnut is not.

Henri Poincaré (1856–1910)

Elected to all five divisions of the French Academy of Sciences

– Geometry

– Mechanics

– Physics

– Geography

– Navigation

From MathWorld

Poincare Conjecture

In its original form, the Poincare conjecture states that every simply connected closed three-manifold is homeomorphic to the three- sphere (in a topologist's sense) , where a three-sphere is simply a generalization of the usual sphere to one dimension higher. More colloquially, the conjecture says that the three-sphere is the only type of bounded three-dimensional space possible that contains no holes. This conjecture was first proposed in 1904 by H. Poincare, and subsequently generalized to the conjecture that every compact n-manifold is homotopy-equivalent to the n-sphere iff it is homeomorphic to the n-sphere. The generalized statement reduces to the original conjecture for n = 3.

The Poincare conjecture has proved a thorny problem ever since it was first proposed, and its study has led not only to many false proofs, but also to a deepening in the understanding of the topology of manifolds (Milnor). One of the first incorrect proofs was due to Poincare himself, stated four years prior to formulation of his conjecture, and to which Poincare subsequently found a counterexample. In 1934, Whitehead proposed another incorrect proof, then discovered a counterexample (the Whitehead link) to his own theorem.

The $n = 2$ case is classical (and was known to 19th century mathematicians), $n = 3$ (the original conjecture) remains open, $n = 4$ was proved by Freedman (1982) (for which he was awarded the 1986 Fields medal), $n = 5$ was demonstrated by Zeeman (1961), $n = 6$ was established by Stallings (1962), and was shown by Smale in 1961 (although Smale subsequently extended his proof to include all).

In April 2002, M. J. Dunwoody produced a five-page paper that purports to prove the conjecture. However, Dunwoody's manuscript was quickly found to be fundamentally flawed (Weisstein 2002). A much more promising result has been reported by Perelman (2002, 2003; Robinson 2003). Perelman's work appears to establish a more general result known as the Thurston's geometrization conjecture, from which the Poincare conjecture immediately follows (Weisstein 2003). Mathematicians familiar with Perelman's work describe it as well thought-out and expect that it will be difficult to locate any substantial mistakes (Robinson 2003).

6. RIEMANN HYPOTHESIS

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called prime numbers, and they play an important role, both in pure mathematics and its applications.

The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function $\zeta(s)$ called the Riemann Zeta function.

The Riemann hypothesis asserts that all interesting solutions of the equation

$$\zeta(s) = 0$$

lie on a straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

In search of grand unified theories.

“ I can safely say that no one understands quantum mechanics”

Richard Feynman (1965)

“ (quantum field theory) a twentieth century scientific theory that uses twenty-first century mathematics”

Edward Witten

7 YANG-MILLS THEORY

The laws of quantum physics stand to the world of elementary particles in the way that Newton’s laws of classical mechanics stand to the macroscopic world.

Almost half a century ago, Yang and Mills introduced a remarkable new framework to describe elementary particles using structures that also occur in geometry.

Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear.

The successful use of Yang-Mills theory to describe the strong interactions of elementary particles depends on a subtle quantum mechanical property called the "mass gap:" the quantum particles have positive masses, even though the classical waves travel at the speed of light. This property has been discovered by physicists from experiment and confirmed by computer simulations, but it still has not been understood from a theoretical point of view.

Progress in establishing the existence of the Yang-Mills theory and a mass gap and will require the introduction of fundamental new ideas both in physics and in mathematics.

More about the RIEMANN HYPOTHESIS

Ask any professional mathematician to name the most important unsolved problem of mathematics and the answer is virtually certain to be, “the Riemann Hypothesis.”

Keith Devlin – The Millennium Problems – 2002

On the Number of Prime Numbers less than a Given Quantity.

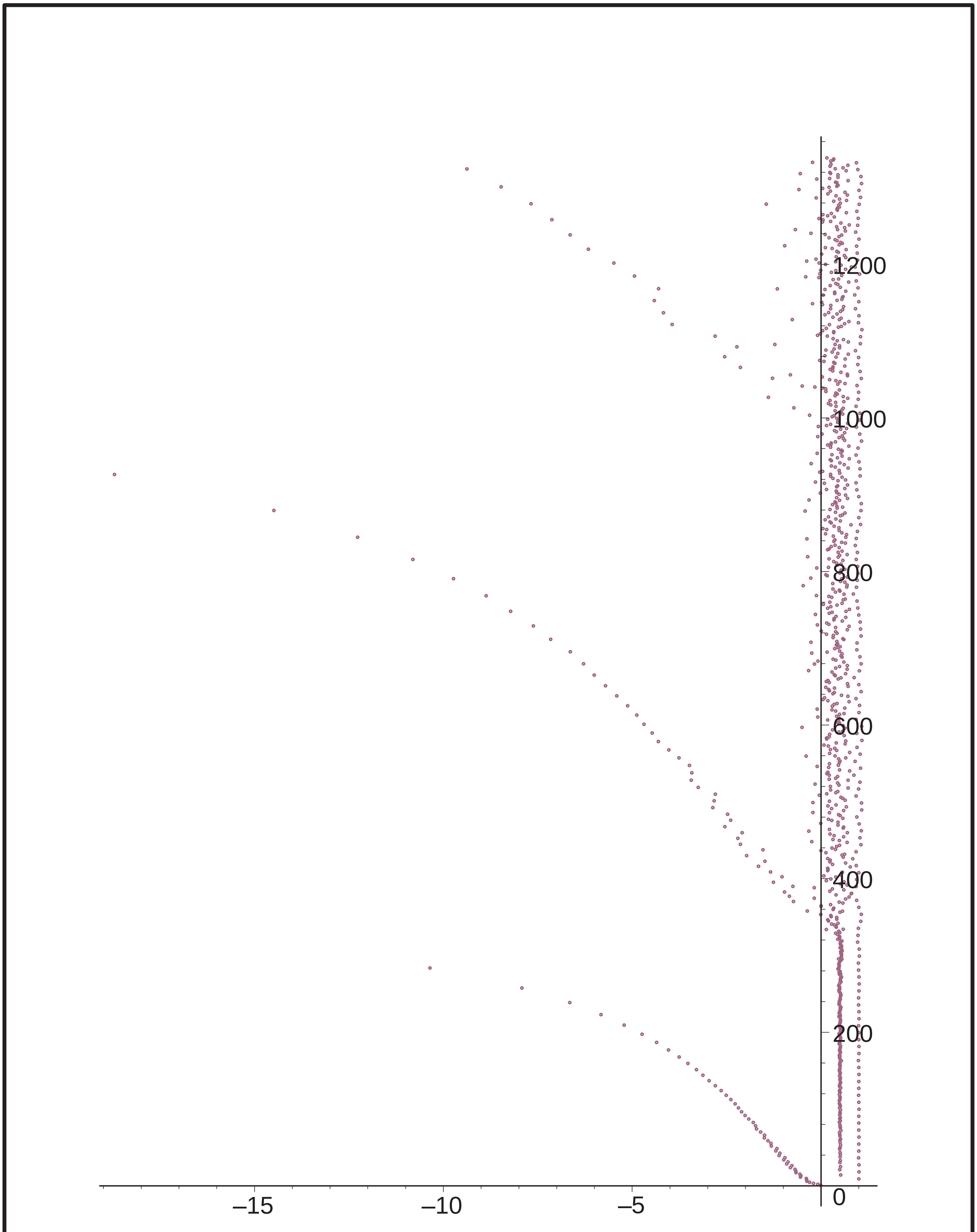
(Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.)

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Translated by David R. Wilkins

" One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation."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The Holy Grail

The Holy Grail in mathematics is the Riemann Hypothesis. Ths problem was formulated in 1859 by Bernard Riemann, one of the extraordinary talents of the 19th century.

The Riemann Hypothesis makes a very precise connection between two seemingly unrelated objects, and if solved, would tell us something profound about the nature of mathematics and, in particular, numbers. Why is the Riemann Hypothesis so important?

Why is it the problem that mathematicians would make a pact with the devil to solve? There are a number of great old unsolved problems in mathematics but none of them have quite the stature of the Riemann Hypothesis – for a variety of reasons both mathematical and sociological.

In common with the other old great unsolved problems, the Riemann Hypothesis is clearly very hard. It has resisted solution for 150 years and has been attempted by many of the greatest minds in mathematics.

David Hilbert one of the seminal figures in mathematical history and a great mathematician re raised the problem at the 1900 International Congress of Mathematics, a conference held every 4 years and the most important international mathematics meeting.

Hilbert, who by that time was the pre eminent mathematician of his generation, raised 23 problems that he thought would shape 20th century mathematics, and in large this proved to be true. This was somewhat self-fulfilling as solving a Hilbert problem was a guarantee of instant fame and perhaps local riches. Many of Hilbert's problems have been now been solved.

The most notable recent example being the Fermat problem solved by Andrew Wiles in 1993–5.

Being one of Hilbert's 23 problems was enough to guarantee the Riemann problem being central. (But there is now also a million dollar bounty in the form of the "Millennium Prize Problem" of the Clay Mathematics Institute of Cambridge.)

Solving one of the great unsolved problems in mathematics is akin to the first ascent of Everest. It is a formidable achievement but after its conquest there is usually nowhere to go but down. Some of the great problems proved to be isolated mountain peaks not connected to any others.

The Riemann Hypothesis is quite different in this regard. There is a large body of mathematics that would instantly become proved if the Riemann Hypothesis was solved. We know many statements of the form "if the Riemann Hypothesis then the following interesting mathematical statement" and this is quite different from the solution of problems such as the Fermat problem. The Riemann Hypothesis can be formulated in many diverse and seemingly unrelated ways, this is one of its beauties. One formulation is that certain numbers related to the so called "Riemann Zeta function" lie in a certain place and this formulation can to some extent be checked numerically.

In one of the largest calculations ever done to date, it was checked that the first hundred billion of these zeros lie where they are supposed to lie. So there are a hundred billion pieces of evidence indicating that the Riemann Hypothesis is true and not a single piece of evidence indicating that it is false.

The average physicist would be overwhelmingly pleased with this much evidence in its favor but to the average mathematician this is hardly evidence at all, merely more than interesting ancillary information.

A proof is required that all of these numbers lie in the right place, not just the first hundred billion, and until the proof is provided the Riemann Hypothesis cannot be incorporated into the corpus of mathematical facts and accepted as true by mathematicians. (Even though it is undoubtably true!)

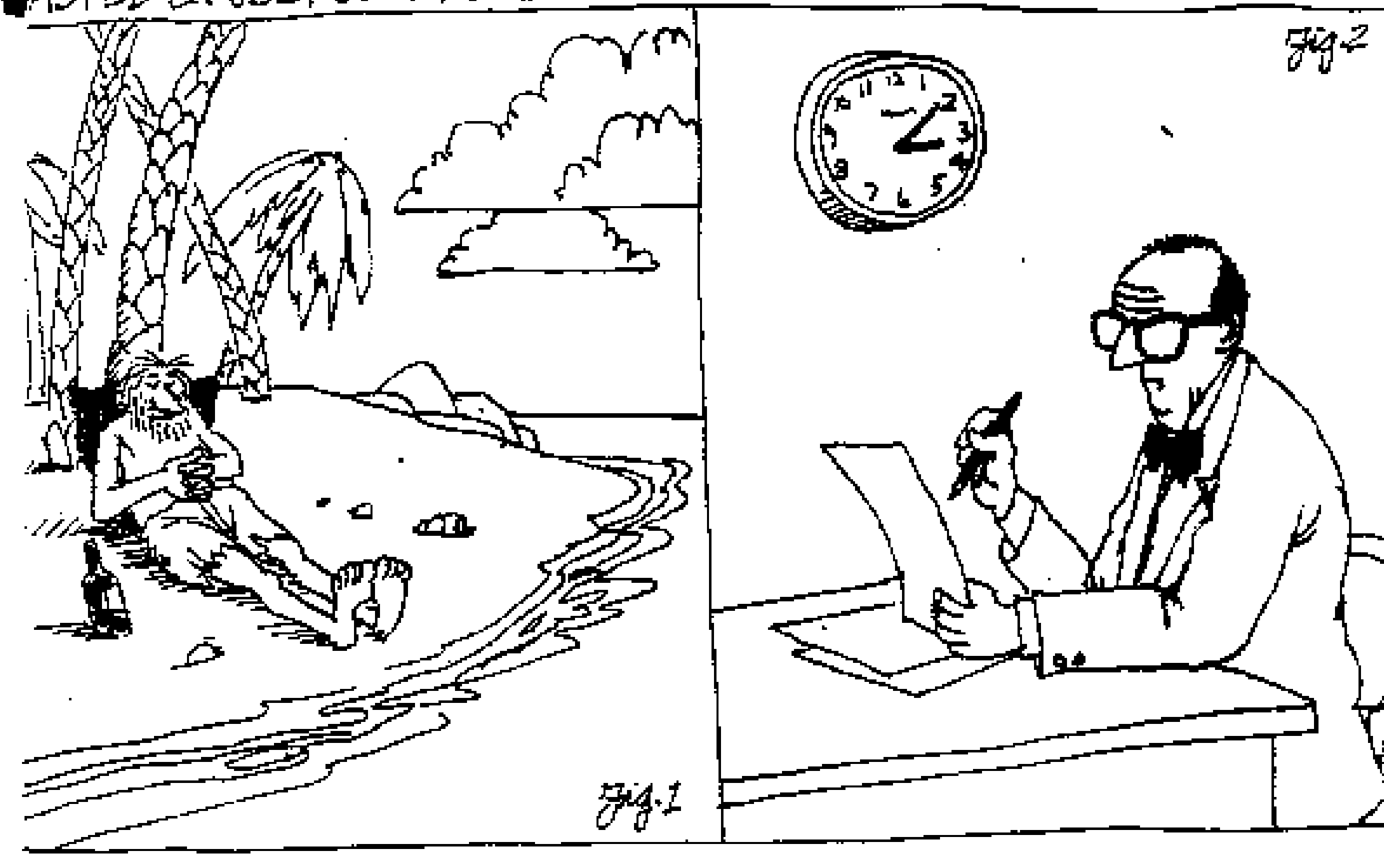
Mathematicians are very fussy about this need for proof. Mathematics is built brick by brick and a single misplaced brick, a single wrong fact, brings the whole edifice down, at least logically. Any statement you like follows logically from a single wrong fact, so in the presence of a single wrong fact, all mathematics is in question.

Accept for a moment that the Riemann Hypothesis is the greatest unsolved problem in mathematics and that the greatest achievement any young graduate student could aspire to is to solve it.

Why isn't it better known? Why hasn't it permeated public consciousness? (The way black holes and unified field theory have, at least to some extent.) Part of the reason for this is it is hard to state precisely.

It requires most of an undergraduate degree in mathematics to be familiar with enough the objects to even accurately state the Riemann Hypothesis. Our suspicion is that only a minority of professional mathematicians – perhaps a quarter – can accurately state the Riemann Hypothesis if asked.

WASTED & USEFUL LIVES



Are grand challenge problems good for mathematics?

Are grand challenge problems good for mathematicians?

Would Nobel prizes in mathematics help or hurt the field?