SOME HIGHLY COMPUTATIONAL PROBLEMS CONCERNING INTEGER POLYNOMIALS OF SMALL NORM.

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The PTE problem is solved by two sets of n integers satisfying any of the following:

$$\sum_{i=1}^{n} \alpha_i^j = \sum_{i=1}^{n} \beta_i^j \quad j = 1, \dots, n-1$$

$$\prod_{i=1}^{n} (x - \alpha_i) - \prod_{i=1}^{n} (x - \beta_i) = C$$

$$(x-1)^n \mid \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i}.$$

The conjecture due to Wright (and others) is that it is always possible.

Almost equivalently (though not quite obviously) find a polynomial with coefficients $\{0, -1, +1\}$ with

$$\|p\|_{L^2\{|z|=1\}} = \sqrt{2n}.$$

Partial History.

- Euler
- Prouhet (1851)
- Tarry (1910) - Small Examples
- Escott (1910) - Small Examples
- Wright, Fuchs (1935) - Easier Waring
- Letac (1941) - Size 9 and 10
- Gloden (1946) - Size 9 and 10
- Smyth (Math Comp. 1991) - Size 10

An **even ideal symmetric solution** of size n is of the form

$$\{\pm\alpha_1, \ldots, \pm\alpha_{n/2}\}, \{\pm\beta_1, \ldots, \pm\beta_{n/2}\}$$

It satisfies

$$\sum_{i=1}^{k/2} \alpha_i^{2j} = \sum_{i=1}^{k/2} \beta_i^{2j} \quad j = 1, \ldots, \frac{k-2}{2}$$

An **odd ideal symmetric solution** of size n is of the form

$$\{\alpha_1, \ldots, \alpha_k\}, \{-\alpha_1, \ldots, -\alpha_k\}$$

and satisfies

$$\sum_{i=1}^{k} \alpha_i^{j} = 0 \quad j = 1, 3, 5, \ldots, k-2$$

Listed below are ideal symmetric solutions for sizes $2 \leq n \leq 10$, the odd symmetric solutions are all perfect. These solutions are listed in abbreviated symmetric form. For example the solution for size 6 is

$$\{\pm 4, \pm 9, \pm 13\}, \{\pm 1, \pm 11, \pm 12\}$$

and the solution for size 5 is

$$\{-8, -7, 1, 5, 9\}, \{8, 7, -1, -5, -9\}.$$

$2: \{3\}, \{1\}$

$3: \{-2, -1, 3\}$

$4: \{3, 11\}, \{7, 9\}$

$5: \{-8, -7, 1, 5, 9\}$

$6: \{4, 9, 13\}, \{1, 11, 12\}$

$7: \{-51, -33, -24, 7, 13, 38, 50\}$

$8: \{2, 16, 21, 25\}, \{5, 14, 23, 24\}$

$9: \{-98, -82, -58, -34, 13, 16, 69, 75, 99\}$

$9: \{-169, -161, -119, -63, 8, 50, 132, 148, 174\}$

$10: \{436, 11857, 20449, 20667, 23750\},$

$\{12, 11881, 20231, 20885, 23738\}$

$10: \{133225698289, 189880696822, 338027122801, 432967471212, 529393533005\},$

$\{87647378809, 243086774390, 308520455907, 441746154196, 527907819623\}$

Size 5. The following is a one parameter example of size 5.

$$F_5 :=$$

$$\left(t+2\,m^2\right)(t-1)\left(t+2\,m^2-1\right)$$

$$\left(t-2\,m^2+1-m\right)\left(t-2\,m^2+m+1\right)$$

$$-\left(t-2\,m^2\right)(t+1)\left(t-2\,m^2+1\right)$$

$$\left(t+2\,m^2-1+m\right)\left(t+2\,m^2-m-1\right)$$

This expands to

$$F_5 := -4\,m^2\,(m-1)\,(2\,m+1)\,(2\,m-1)$$

$$(m+1)\left(2\,m^2-1\right)$$

Size 6. It is possible to completely solve the even symmetric problem of size 6 in Maple. Basically one just uses "solve". it gives the following general solution (translated with $a_6 = 0$.)

This gives as rational solution of size 6:

$$\{a_2 = a_2, b_1 = b_1, b_3 = b_3,$$

$$b_2 = \frac{{a_2}^2 - a_2 b_3 + b_1 b_3 - a_2 b_1}{-b_1 - b_3 + a_2},$$

$$a_1 = 2/3\, \frac{{a_2}^2 - {b_3}^2 - {b_1}^2 - b_1 b_3}{-b_1 - b_3 + a_2},$$

$$a_3 = \frac{-{b_1}^2 - b_1 b_3 + a_2 b_1 + a_2 b_3 - {b_3}^2}{-b_1 - b_3 + a_2}\}$$

The following is a simple two parameter example of size 6.

$$F_6 :=$$

$$\left(t^2 - (2\,n + 2\,m)^2\right)\left(t^2 - (nm + n + m - 3)^2\right)$$

$$\left(t^2 - (nm - n - m - 3)^2\right)$$

$$-\left(t^2 - (2\,n - 2\,m)^2\right)\left(t^2 - (-nm + n - m - 3)^2\right)$$

$$\left(t^2 - (-nm - n + m - 3)^2\right)$$

On expansion one sees that

$$F_6 := -16\,nm\,(m-1)\,(m+3)\,(m-3)\,(m+1)$$

$$(n-1)\,(n+3)\,(n-3)\,(n+1)$$

Size 7. Gloden simplified.

$$(t - R_1)(t - R_2)(t - R_3)(t - R_4)$$
$$(t - R_5)(t - R_6)(t - R_7)$$
$$-(t + R_1)(t + R_2)(t + R_3)(t + R_4)$$
$$(t + R_5)(t + R_6)(t + R_7)$$

where

$$R_1 := -\left(-3\,j^2k + k^3 + j^3\right)\left(j^2 - kj + k^2\right)$$
$$R_2 := (j + k)(j - k)\left(j^2 - 3\,kj + k^2\right)j$$
$$R_3 := (j - 2\,k)\left(j^2 + kj - k^2\right)kj$$
$$R_4 := -(j - k)\left(j^2 - kj - k^2\right)(-k + 2\,j)\,k$$
$$R_5 := -(j - k)\left(-2\,kj^3 + j^4 - j^2k^2 + k^4\right)$$
$$R_6 := \left(j^4 - 4\,kj^3 + j^2k^2 + 2\,k^3j - k^4\right)k$$
$$R_7 := \left(j^4 - 4\,kj^3 + 5\,j^2k^2 - k^4\right)j$$

On expansion

$$F_7 = 2\,j^3k^3\,(-k+2\,j)\,(j-2\,k)\,(j+k)$$
$$\left(j^2+kj-k^2\right)\left(j^2-kj-k^2\right)\left(j^2-3\,kj+k^2\right)$$
$$\left(-3\,j^2k+k^3+j^3\right)\left(j^4-4\,kj^3+5\,j^2k^2-k^4\right)$$
$$\left(-2\,kj^3+j^4-j^2k^2+k^4\right)(j-k)^3$$
$$\left(j^4-4\,kj^3+j^2k^2+2\,k^3j-k^4\right)\left(j^2-kj+k^2\right)$$

For example with $j := 2$ and $k := 3$

$$(t-7)\,(t-50)\,(t+24)\,(t+33)$$
$$(t-13)\,(t+51)\,(t-38)$$
$$-\,(t+7)\,(t+50)\,(t-24)\,(t-33)$$
$$(t+13)\,(t-51)\,(t+38)$$
$$= 13967553600$$

Size 8. A (homogenous) size 8 solution due to Chernick

$$F_8 := \left(t^2 - {R_1}^2\right)\left(t^2 - {R_2}^2\right)\left(t^2 - {R_3}^2\right)\left(t^2 - {R_4}^2\right)$$

$$- \left(t^2 - {R_5}^2\right)\left(t^2 - {R_6}^2\right)\left(t^2 - {R_7}^2\right)\left(t^2 - {R_8}^2\right)$$

where

$$R_1 := 5\,m^2 + 9\,mn + 10\,n^2$$

$$R_2 := m^2 - 13\,mn - 6\,n^2$$

$$R_3 := 7\,m^2 - 5\,mn - 8\,n^2$$

$$R_4 := 9\,m^2 + 7\,mn - 4\,n^2$$

$$R_5 := 9\,m^2 + 5\,mn + 4\,n^2$$

$$R_6 := m^2 + 15\,mn + 8\,n^2$$

$$R_7 := 5\,m^2 - 7\,mn - 10\,n^2$$

$$R_8 := 7\,m^2 + 5\,mn - 6\,n^2$$

Size 9. We know no parametric solution of size 9. Indeed only two solutions are known. Both are symmetric and they are the following

$$[-98, -82, -58, -34, 13, 16, 69, 75, 99]$$

and

$$[174, 148, 132, 50, 8, -63, -119, -161, -169]$$

Size 10. The following size 10 example is due to Letac (and Smyth)

$$F_{10} :=$$

$$\left(t^2 - {R_1}^2\right)\left(t^2 - {R_2}^2\right)\left(t^2 - {R_3}^2\right)$$

$$\left(t^2 - {R_4}^2\right)\left(t^2 - {R_5}^2\right)$$

$$-\left(t^2 - {R_6}^2\right)\left(t^2 - {R_7}^2\right)\left(t^2 - {R_8}^2\right)$$

$$\left(t^2 - {R_9}^2\right)\left(t^2 - {R_{10}}^2\right)$$

where

$$R_1 := (4\,n + 4\,m)$$

$$R_2 := (mn + n + m - 11)$$

$$R_3 := (mn - n - m - 11)$$

$$R_4 := (mn + 3\,n - 3\,m + 11)$$

$$R_5 := (mn - 3\,n + 3\,m + 11)$$

$$R_6 := (4\,n - 4\,m)$$

$$R_7 := (-mn + n - m - 11)$$

$$R_8 := (-mn - n + m - 11)$$

$$R_9 := (-mn + 3\,n + 3\,m + 11)$$

$$R_{10} := (-mn - 3\,n - 3\,m + 11)$$

On expansion

$$F_{10} := c_0 + c_2 t^2 + c_4 t^4 + c_6 t^6$$

And each coeffecient except c_0 has a factor

$$m^2 n^2 - 13\,n^2 + 121 - 13\,m^2$$

So any solution of the above biquadratic gives a size 10 solution. For example:

$n := 153/61$ and $m = 191/79$

$n := -296313/249661$ and $m = -1264969/424999$