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Littlewood's 22nd Problem

"If the n_m are integral and all different, what is the lower bound on the number of real zeros of

$$\sum_{m=1}^{N} \cos(n_m\theta)??$$

Possibly $N-1$, or not much less."

Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" poses this. No progress appears to have been made on this in the last half century. Until now.

Theorem 1 *It is possible to construct cosine polynomials with the n_m integral and all different, so that the number of real zeros of*

$$\sum_{m=1}^{N} \cos(n_m\theta)$$

is

$$O\left(N^{9/10}\right).$$

We also prove in a positive direction.

Denote the number of zeros of T in the period $[-\pi, \pi)$ by $N(T)$.

Theorem 2 *Suppose the set* $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ *is finite and the set* $\{j \in \mathbb{N} : a_j \neq 0\}$ *is infinite. Let*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,.$$

Then

$$\lim_{n \to \infty} N(T_n) = \infty\,.$$

One of our main tools for this, not surprisingly, is the resolution of the Littlewood Conjecture.

The next two results SHOULD be straightforward corollaries of the above result (????)

Theorem 3 *Let A_N denote the the lower bound on the number of zeros in period $[-\pi, \pi)$ of all N term cosine sums of the form*

$$\sum_{m=1}^{N} \cos(n_m \theta)$$

then

$$\lim_{n \to \infty} B_n = \infty.$$

As an answer to a question of B. Conrey.

Theorem 4 *Let B_N denote the the lower bound on the number of zeros in period $[-\pi, \pi)$ of all N term cosine sums of the form*

$$\sum_{m=1}^{N} \pm \cos(n\theta)$$

then

$$\lim_{n \to \infty} A_n = \infty \, .$$

Lemma 1 *Let* $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ *be nonnegative integers and let*

$$S_m(t) = \sum_{j=0}^{m} A_j \cos(\lambda_j t) \ .$$

Then

$$\int_{-\pi}^{\pi} |S_m(t)|\, dt \geq \frac{1}{80} \sum_{j=0}^{m} \frac{|A_{m-j}|}{j+1} \, .$$

Littlewood's 22nd Problem

Problem 1 *"If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m\theta)$? Possibly $N-1$, or not much less."*

In terms of reciprocal polynomials one is looking for a reciprocal polynomial with coefficients 0 and 1 with 2n terms and n-1 or fewer zeros.

Even achieving n-1 is fairly hard.

An exhaustive search up to $2n = 32$ yielded only the 2 examples below with n-1 zeros of modulus one and none with n-2 or fewer zeros.

There where only 11 more examples with exactly n zeros. It is hard to see how one might generate infinitely many examples or indeed why Littlewood made his conjecture.

$x^{27} + x^{26} + x^{25} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^2 + x + 1$

and

$x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{17} + x^{14} + x^{12} + x^{11} + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

The following is a reciprocal polynomial with 32 terms and exactly 14 zeros of modulus 1. So it corresponds to a cosine sum of 16 terms with 14 zeros in $[-\pi, \pi)$. In other words the sharp version of Littlewood's conjecture is false. (Though barely.)

$1 + x + x^2 + x^4 + x^3 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{18} + x^{20} + x^{22} + x^{23} + x^{24} + x^{25} + x^{26} + x^{29} + x^{30} + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} + x^{37} + x^{38}$

The following is a reciprocal polynomial with 280 terms and 52 zeros of modulus 1. So it corresponds to a cosine sum of 140 terms with 52 zeros in $[-\pi, \pi)$. In other words the sharp version of Littlewood's conjecture is false. Though this time by a margin.

It was found by a version of the greedy algorithm (and some guessing). There is no reason to believe it is a minimal example.

$$(1 + x + x^2 + x^4 + x^3 + x^5 + x^6 + x^7 + x^8 + x^9 +
x^{10} + x^{11} + x^{12} + x^{13} + x^{19} + x^{14} + x^{15} + x^{17} +
x^{18} + x^{16} + x^{20} + x^{21} + x^{22} + x^{23} + x^{24} + x^{25} +
x^{26} + x^{27} + x^{28} + x^{29} + x^{30} + x^{31} + x^{32} + x^{33} +
x^{34} + x^{35} + x^{36} + x^{37} + x^{38} + x^{39} + x^{40} + x^{41} +
x^{42} + x^{43} + x^{44} + x^{45} + x^{46} + x^{47} + x^{48} + x^{49} +
x^{50} + x^{51} + x^{52} + x^{53} + x^{54} + x^{55} + x^{56} + x^{57} +
x^{58} + x^{59} + x^{60} + x^{61} + x^{62} + x^{63} + x^{64} + x^{65} +
x^{66} + x^{67} + x^{68} + x^{69} + x^{70} + x^{71} + x^{72} + x^{73} +
x^{74} + x^{75} + x^{76} + x^{78} + x^{79} + x^{80} + x^{81} + x^{82} +$$

$$
x^{83} + x^{77} + x^{84} + x^{85} + x^{86} + x^{87} + x^{88} + x^{89} +
x^{90} + x^{91} + x^{92} + x^{93} + x^{94} + x^{95} + x^{96} + x^{97} +
x^{98} + x^{99} + x^{100} + x^{101} + x^{102} + x^{103} + x^{104} +
x^{105} + x^{106} + x^{107} + x^{108} + x^{109} + x^{110} + x^{111} +
x^{112} + x^{113} + x^{114} + x^{115} + x^{116} + x^{117} + x^{118} +
x^{119} + x^{120} + x^{121} + x^{122} + x^{123} + x^{129} + x^{130} +
x^{131} + x^{132} + x^{133} + x^{135} + x^{136} + x^{137} + x^{138} +
x^{139} + x^{140} + x^{142} + x^{144} + x^{146} + x^{149} + x^{150} +
x^{154} + x^{155} + x^{158} + x^{160} + x^{162} + x^{164} + x^{165} +
x^{171} + x^{166} + x^{167} + x^{169} + x^{168} + x^{172} + x^{173} +
x^{174} + x^{175} + x^{181} + x^{182} + x^{183} + x^{184} + x^{185} +
x^{186} + x^{187} + x^{188} + x^{189} + x^{190} + x^{191} + x^{192} +
x^{193} + x^{194} + x^{195} + x^{196} + x^{197} + x^{198} + x^{199} +
x^{200} + x^{201} + x^{202} + x^{203} + x^{204} + x^{205} + x^{206} +
x^{207} + x^{208} + x^{209} + x^{210} + x^{211} + x^{212} + x^{213} +
x^{214} + x^{215} + x^{216} + x^{217} + x^{218} + x^{219} + x^{220} +
x^{221} + x^{222} + x^{223} + x^{224} + x^{225} + x^{226} + x^{227} +
x^{228} + x^{230} + x^{231} + x^{232} + x^{233} + x^{234} + x^{235} +
x^{229} + x^{236} + x^{237} + x^{238} + x^{239} + x^{240} + x^{241} +
x^{242} + x^{243} + x^{244} + x^{245} + x^{246} + x^{247} + x^{248} +
x^{249} + x^{250} + x^{251} + x^{252} + x^{253} + x^{254} + x^{255} +
$$

$$x^{256}+x^{257}+x^{258}+x^{259}+x^{260}+x^{261}+x^{262}+$$
$$x^{263}+x^{264}+x^{265}+x^{266}+x^{267}+x^{268}+x^{269}+$$
$$x^{270}+x^{271}+x^{272}+x^{273}+x^{274}+x^{275}+x^{276}+$$
$$x^{277}+x^{278}+x^{279}+x^{280}+x^{281}+x^{282}+x^{283}+$$
$$x^{284}+x^{285}+x^{286}+x^{287}+x^{288}+x^{289}+x^{290}+$$
$$x^{291}+x^{292}+x^{293}+x^{294}+x^{295}+x^{296}+x^{297}+$$
$$x^{298}+x^{299}+x^{300}+x^{301}+x^{302}+x^{303}+x^{304})$$

Auxiliary Functions

The key is to construct n term cosine sums that are large most of the time.

Lemma 2 *There is a constant C such that for all n and $\alpha > 1$ there is a sequence $a_0, \ldots, a_n$ with each $a_i \in \{0, 1\}$ such that*

$$\textit{meas}\{t \in [-\pi, \pi) : |P_n(t)| \leq \alpha\} \leq C\alpha n^{-1/2}.$$

where

$$P_n(t) = \sum_{j=0}^{n} a_j \cos(jt).$$

The Main Theorem

Theorem 5 *It is possible to construct cosine polynomials with the* n_m *integral and all different, so that the number of real zeros of*

$$\sum_{m=1}^{N} \cos(n_m \theta)$$

is

$$O\left(N^{9/10} \log^{1/5}(N)\right).$$

The proof follows immediately from the following lemma and Lemma 2.

(Take $m := N + 1$, $n = m^{2/5} \log^{-4/5}(m)$, $\alpha = n^{1/4}$ and $\beta = C\alpha n^{-1/2} = Cn^{-1/4}$.)

Lemma 3 *Let $m \le n$,*

$$D_m(t) := \sum_{j=0}^{m} \cos(jt)\,,$$

$$P_n(t) := \sum_{j=0}^{n} a_j \cos(jt)\,, \qquad a_j \in \{0,1\}\,.$$

Suppose $\alpha \ge 1$ and

$$\text{meas}\{t \in [-\pi,\pi) : |P_n(t)| \le \alpha\} \le \beta\,.$$

Let $S_m := D_m - P_n$. Then the number of zeros of S_m in $[-\pi,\pi)$ is at most

$$\frac{c_1 m}{\alpha} + c_2 m\beta + c_3 n m^{1/2} \log m\,,$$

where c_1, c_2, and c_3 are absolute constants.

For this we need the following consequence of the Erdős-Turán Theorem.

Lemma 4 *Let*

$$S_m(t) = \sum_{j=0}^{n} a_j \cos(jt)\,, \qquad a_j \in \{0, 1\}\,.$$

Denote the number of zeros of S_m *in* $[\alpha, \beta] \subset [-\pi, \pi)$ *by* $N([\alpha, \beta])$*. Then*

$$N([\alpha, \beta]) \le c_4 m(\beta - \alpha) + c_4 \sqrt{m} \log m\,,$$

where c_4 *is an absolute constant.*

Average Number of Zeros

Lemma 5 *Suppose that p is a polynomial of degree exactly n and p has k zeros of modulus greater than 1 and j zeros of modulus 1 then for any m*

$$\Big(z^m p(z) \pm p^*(z)\Big)$$

has degree m+n and at least m+n - 2k roots of modulus 1.

Proof. Rouché's theorem shows that

$$(1+\epsilon) z^m p(z) \pm p^*(z)$$

and

$$z^m p(z)$$

have the same number of roots inside the unit disk. Note that $|p(z)| = |p^*(z)|$ for $|z| = 1$.

So with $\epsilon = 0$, $z^m p(z) \pm p^*(z)$ has all but k zeros in the closed unit disk.

Now use the fact that $z^m p(z) \pm p^*(z)$ is reciprocal so has the same number of zeros of modulus less than 1 as of modulus greater than 1.

Lemma 6 *Suppose that p is a polynomial of degree exactly n and $p(0) \neq 0$. Consider*

$$P := \left(z^m p(z) \pm p^*(z)\right)$$

and

$$Q := \left(z^m p^*(z) \pm p(z)\right).$$

with the same choice of sign (ie the cos case and the sin case). Suppose P has j_1 zeros of modulus 1 and Q has j_2 zeros of modulus 1. Then

$$j_1 + j_2 \geq 2m.$$

Proof. Use the previous lemma and note that if p has k zeros of modulus greater than 1 and j zeros of modulus 1 then p^* has $n-k-j$ zeros of modulus greater than 1 and j zeros of modulus 1.

Note that if $M := (m-n)/2 \geq 1$ with M an integer then

$$C := \sum_{i=M}^{n+M} a_i \cos it$$

and

$$S := \sum_{i=M}^{n+M} a_i \sin it$$

correspond to

$$P(z) := \Big(z^m p(z) \pm p^*(z)\Big)$$

with

$$p(z) = \sum_{i=0}^{n} a_i z^i.$$

Also zeros of P of modulus 1 correspond (with the same count) to zeros of the trigonometric polynomials C amd S in the period $[0, \pi)$.

Lemma 7 *Suppose $a_{n+M} \neq 0$. Consider*

$$C(t) := \sum_{i=M}^{n+M} a_i \cos it$$

and

$$C^*(t) := \sum_{i=M}^{n+M} a_{(n+M+M-i)} \cos it$$

which reverses the coefficients.

Let w_1 be the number of zeros of C in the period $[0, \pi)$ and let w_2 be the number of zeros of C^* in the period $[0, \pi)$ then

$$w_1 + w_2 \geq m \geq n + 1.$$

Furthermore $w_1 \geq m$ and $w_2 \geq m$.

Averaging over any reasonable class of sums gives:

Lemma 8 *The average number of zeros over the classes*

$$\Big\{\sum_{i=1}^{n} \pm \cos it\Big\}$$

and

$$\Big\{\sum_{i=1}^{n} \delta_i \cos it,\ \ \delta_i \in 0, 1\Big\}$$

is at least n/2.