A Question of Smale

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2005

Abstract

Abstract: Two problems of Smale that imply a version of $P \neq NP$ over $\mathbb{C}$.

These are from a paper of Smale entitled ”Mathematical problems for the next century.” They are problem 4.

Introduction

Smale Verbatim

Problem 4: Integer zeros of a polynomial of one variable.

A program for a polynomial $f \in \mathbb{Z}[t]$ of one variable with integer coefficients is the object $(1, t, u_1, ..., u_k)$ where $u_k = f$, and for all l, $u_l = u_i \circ u_j$, $i, j < l$ and $\circ$ is one of $+$,$-$ or $*$, Here $u_0 = t$ and $u_{-1} = 1$. Then $\tau(f)$ is the minimum of k over all such programs.

Is the number of distinct integer zeros of f, polynomially bounded by $\tau(f)$? In other words, is

$$Z(f) \leq a\tau(f)^c \, for \, all f \in \mathbb{Z}[t].$$

Here Z(f) is the number of distinct integer zeros of f with a, c universal constants.

From earlier results of Strassen, communicated via Schönhage, Shub, and Bürgisser, it follows that the exponent c has to be at least 2.

Mike Shub and I discovered this problem in our complexity studies. We proved that an affirmative answer implied the intractibility of the Nullstellensatz as a decision problem over $\mathbb{C}$ and thus $P \neq NP$ over $\mathbb{C}$.

See (Shub-Smale, 1995) and also BCSS.

Since the degree of f is less than or equal to 2^τ, $\tau = \tau f)$, there are no more than 2^τ zeros altogether.

For Chebyshev polynomials, the number of distinct real zeros grows exponentially with τ.

Many of the classic diophantine problems are in two or more variables. This problem asks for an estimate in just one variable, and nevertheless seems not so easy.

Here is a related problem. A program for an integer m is the object $(1, m_1, ..., m_l)$ where $m_k = m$, $m_0 = 1$ and for all l, $m_l = m_j \circ m_k$, $j, k < l$ and $\circ$ is one of $+$,$-$ or $*$, Then $\tau(f)$ is the minimum of k over all such programs. Thus $\tau(m)$ represents the shortest way to build up an integer m starting from 1 using plus, minus, and times.

Problem: Is there a constant c such that $\tau(k!) \leq (\log k)^c$ for all integers k?

One might expect this to be false, so that k! is hard to compute (see Shub-Smale, 1995).

A Start

As above let S_k be the set of all polynomials in $\mathbb{Z}[t]$ for which there exists a program

$$(1, t, u_1, ..., u_k)$$

where $u_k = f$, and for all l,

$$u_l = u_i \circ u_j,\ i, j < l$$

and $\circ$ is one of $+$,$-$ or $*$,

Here $u_0 = t$ and $u_{-1} \in A$ where A is a fixed finite set of integers.

Some programs with $k := 1$

$$[1, t, 2], [1, t, 0], [1, t, 1], [1, t, t + 1],$$

$$[1, t, -t+1], [1, t, t], [1, t, t-1], [1, t, 2\,t], [1, t, t^2]$$

A variety of length 3 programs are:

$$[1, t, 2\,t, -t + 1, -t^2 + t]$$

$$[1, t, 0, 2\,t, t]$$

$$[1, t, t + 1, 2\,t, 2]$$

$$[1, t, t - 1, -t + 1, -2\,t + 1]$$

$$[1, t, t^2, 2\,t^2, 4\,t^4]$$

One can compute that S_1 has cardinality 21, S_2 has cardinality 189 and S_3 has cardinality 11301.

Theorem 1

$$|S_k| \leq (3/2)^k (k+2)!(k+1)! \leq 2^{ck \log(k)}$$

for some absolute constant c.

Proof. This is an easy induction. Every program of length k gives rise to at most $3((k+2)(k+1)/2)$ programs of length $k+1$.

A Math Review of de A. Moreira (1997)

The cost of an integer n, denoted $\tau(n)$, is the least integer m such that there exists a sequence $(s_0, \cdots, s_m)$ of integers with the property that $s_0 = 1$, $s_m = n$, and for each $l \in \{1, \cdots, m\}$ there exist $i, j \in \{0, \cdots, l-1\}$ such that $s_l = F(s_i, s_j)$ where F is one of addition, subtraction, or multiplication.

In this paper, the author proves that for any $\epsilon > 0$, the ratio

$$\frac{\#\{k \leq n \,\big|\, \tau(k) \geq \frac{\log k}{\log\log k} + (1-\epsilon)\frac{\log k \log\log\log k}{(\log\log k)^2}\}}{n}$$

approaches 1 as n goes to infinity and that for any $\epsilon > 0$,

$$\tau(n) \leq \frac{\log n}{\log\log n} + (3+\epsilon)\frac{\log n \log\log\log n}{(\log\log n)^2}$$

for n sufficiently large. He concludes that

$$\tau(n) \geq \log n / \log\log n$$

for almost all n, and that for any $\epsilon > 0$, if n is large enough, then

$$\tau(n) \leq (1+\epsilon)\log n / \log\log n.$$

These bounds improve on earlier results, such as those in a paper by W. de Melo and B. F. Svaiter **??**Proc. Amer. Math. Soc. 124 (1996), no. 5, 1377–1378; MR1307510 (96g:11150)]. The author also extends the notion of cost to a polynomial in several variables with integral coefficients and establishes bounds for this case which are analogous to those given above for the integer cost function.

The final section of the paper is devoted to a discussion of some open problems relating to the integer cost function τ. The most significant of these is to determine whether there exist a sequence of positive integers $\{n_i\}_{i=0}^{\infty}$ and a positive constant c such that $\tau(n_i i!) \leq (\log i)^c$ for all i. In a paper by M. Shub and S. Smale **??**Duke Math. J. 81 (1995), no. 1, 47–54 (1996)] it is shown that if no such sequence exists, then an algebraic version of NP $\neq$ P is true.

A Math Review of Q. Cheng (2004)

It has long been observed that certain factorization algorithms provide a way to write products of large numbers of integers succinctly. In this paper, we study the problem of representing the product of all integers from 1 to n ($n!$) by straight-line programs.

Formally, we say that a sequence of integers a_n is ultimately $f(n)$-computable if there exists a nonzero integer sequence m_n such that for any $n, a_n m_n$

can be computed by a straight-line program (using only additions, subtractions and multiplications) of length at most $f(n)$. M. Shub and S. Smale [Duke Math. J. 81 (1995), no. 1, 47–54 (1996); MR1381969 (97h:03067)] showed that if $n!$ is ultimately hard to compute, then the algebraic version of NP $\neq$ P is true.

Assuming a widely believed number theory conjecture concerning the density of smooth numbers in short intervals, a subexponential upper bound

$$(\exp(c\sqrt{\log n \log\log n}))$$

for the ultimate complexity of $n!$ is proved in this paper, and a random subexponential algorithm constructing a correspondingly short straight-line program is presented as well.

Binomial Coefficients with Division:

Let N_k be the set of all integers m for which there exists a program $(0, 1, u_1, ..., u_k)$ where $u_k = m$ and for all l,

$$u_l = u_i \pm u_j,\; i, j < l$$

or

$$u_l = u_i * u_j,\; i, j < l$$

or

$$u_l = u_i / u_j,\; i, j < l.$$

(Here division is "integer division" so $a/b := q$ where $a = bq + r$ where $0 \leq r < b$.)

Theorem 2 *Both*

$$\binom{n}{m}$$

and

$$n!$$

are in

$$N_{O(\log(n))}.$$

Proof

Consider

$$F_n := (1 + 10^n)^n$$

Then F_n is an integer with n^2 digits with $\binom{n}{m}$ embedded in the middle. Note that $\binom{n}{m}$ is of maximum size 2^n. So in the expansion each binomial is seperatd by zeros.

So, for example,

$$(1 + 10^8)^8 =$$

$$1000000080000002800000056000000$$

$$7000000056000000280000000$$

$$800000001.$$

Note that F_n evaluates as a straight line program in in $O(\log(n)$ steps.

Note also that

$$(2^n)! = \binom{2^n}{2^{n-1}}\binom{2^{n-1}}{2^{n-2}}^2\binom{2^{n-2}}{2^{n-3}}^4\cdots$$

So if binomial coefficients are polynomial in $\log(n)$ then so are factorials (or at least factorials of powers of 2).

So if spliting an integer in half is $O(\log(n))$ then so is computing factorials.

Division allows for this as follows. To get the (m+1)th through nth digits of M compute as follows. Let

$$a := (m/10^m) * 10^m.$$

Then a has lowest m digits zero and the rest agree with those of M. Now let

$$b := (n/10^n) * 10^n$$

and consider $(b - a)/10^m$.

Questions and stuff.

Question. Does anything distinguish S_k analytically/combinatorially?

Question. Can one tell whether an element is in S_k without computing all of S_k?

Question. Is number of irreducible factors the right question?

Question. Find an unconditional algorithm for $n!$ that is subexpontial in the sense of [**?**]. That is better than $e^{\sqrt{\log n}}$. And does not require assumptions on smooth numbers.

Question. What about binomial coefficients?

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