## 1 Spectra of Pisot and Salem Numbers.

For $q \in (1,2)$ define

$$\Lambda(q) := \{\epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n : \epsilon_i \in \{-1,0,1\}\}$$

$$A(q) := \{\epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n : \epsilon_i \in \{-1,1\}\}.$$

Further define

$$l(q) = \inf\{(|y| : y \in \Lambda(q), y \neq 0\}$$

$$a(q) = \inf\{|y| : y \in A(q), y \neq 0\}.$$

Erdős, Joó and Komornik in 1990 ask about $l(q)$ for Pisot numbers $q$.

Our main purpose is to give a general method for computing $l(q)$.

A related question is for which non-Pisot numbers (if any) is $a(q)$ non-zero. We give an infinite class of Salem numbers where $a(q) \neq 0$.

It is clear that $A(q) \subset \Lambda(q)$. Results known for $A(q)$ specifically are due to Peres and Solomyak. They show that $A(q)$ is dense in $\mathbb{R}$ for a.e. $q \in (\sqrt{2}, 2)$. Also if $q \in (1, \sqrt{2})$ and $q^2$ is not the root of a height 1 polynomial, then $A(q)$ is dense.

We give some examples of non Pisot q where $A(q)$ is discrete. The existence of such a $q$ is somewhat surprising, because the analog, $\Lambda(q)$, is thought to be discrete if and only if $q$ is Pisot. Though it is still open if there exists a $q$ that is not Pisot with $l(q) > 0$.

If $q$ does not satisfy a polynomial of height 1, then $l(q) = 0$ by a Pigeon hole argument.

If $q$ is the Pisot number that satisfies $q^3 - q^2 - 1$, then $l(q) = q^2 - 2$ (Komornik et al)

If $q$ is the Pisot number satisfying $q^n - q^{n-1} - \cdots - 1$ then $l(q) = \frac{1}{q}$ (Erdős, Joo and Joo).

We give an algorithm that calculates $l(q)$ for any Pisot number $q$. This is similar to an algorithm of Ka-Sing Lau.

Another question that can be examined with this algorithm asks for which Pisot numbers $q$ does $q$ satisfy a polynomial with $\pm 1$ coefficients.

Surprisingly there exists Pisot numbers which do not satisfy such a polynomial. For the Pisot numbers that do satisfy a $\pm 1$ polynomial, it is often possible to use LLL to find this polynomial.

**Basis of the Algorithm.**

**Lemma 1.** *Let $S$ be a finite set of integers. Let $p(x)$ be a degree $n$ polynomial with coefficients in $S$. Let $s_l \leq S \leq s_u$, be lower and upper bounds for the integers in $S$, and let $q > 1$. Denote $\alpha_u := \frac{-s_l}{q-1}$ and $\alpha_l := \frac{-s_u}{q-1}$. If $p(q) \notin [\alpha_l, \alpha_u]$ then $q \cdot p(q) + s \notin [\alpha_l, \alpha_u]$ for all $s \in S$.*

**Lemma 2.** *If $q$ is a Pisot number, and $S$ is a finite set of integers, then $|\Lambda_S(q) \cap [a, b]|$ is finite.*

**Programming Tricks.**

When calculating $\Lambda(q)$ a list of all polynomials examined must be kept. To reduce the space requirements, the actual code stores the remainders modulo the minimal polynomial of $q$. This keeps all the calculations exact.

This has a second advantage. If $p(q) = p'(q)$ where $p$ and $p'$ are polynomials, then the remainders of $p$ and $p'$ when divided by the minimal polynomial of $q$ are equal. Thus duplication within the spectrum is easily recognized.

With a careful implementation, this technique can be parallelized.

**Questions on $\Lambda(q)$ and $A(q)$ for Pisot numbers $q$.**

We calculated $l(q)$ for all Pisot numbers up to and including degree 9, and $a(q)$ for Pisot numbers up to and including degree 10 (using David Boyd's algorithm).

There are 232 Pisot numbers of degree less than or equal to 10 (between 1 and 2 inclusive).

The number of Pisot numbers of degree less than or equal to 9 where $a(q) = l(q)$ is fairly small, and it would be interesting to know if the set of Pisot numbers with this property is finite.

The next question of interest is which Pisot numbers $q$ satisfy a $\pm 1$ polynomial. This is equivalent to asking if $0 \in A(q)$. This was answered in the affirmative for most Pisot numbers. The first failure is of degree 6. It is interesting to note that all of the examples found where $0 \notin A(q)$ are such that this root $q$ is greater than 1.95.

In the largest calculations we did the spectrum sizes exceeded 20 million

A spectrum of size over about 48 million broke our progam and accounted for 16 out of the 232 Pisot numbers of degree 10 or less.

| Pisot polynomial | $q$ | $l(q)$ | Polynomial associated with $l(q)$ |
|---|---|---|---|
| $x^2 - x - 1$ | 1.618034 | 0.618034 | $x - 1$ |
| $x^3 - 2x^2 + x - 1$ | 1.754878 | 0.245122 | $x - 2$ |
| $x^4 - x^3 - 1$ | 1.380278 | 0.008993 | $x^3 - 4x^2 + 5$ |
| $x^4 - 2x^3 + x - 1$ | 1.866760 | 0.13324 | $x - 2$ |
| $x^4 - x^3 - 2x^2 + 1$ | 1.905166 | 0.068706 | $x^3 - 3x^2 + x + 2$ |
| $x^5 - x^4 - x^3 + x^2 - 1$ | 1.443269 | 0.002292 | $4x^2 - 3x - 4$ |
| $x^5 - x^3 - x^2 - x - 1$ | 1.534158 | 0.002155 | $2x^4 - 3x^3 + x^2 - 3x + 2$ |
| $x^5 - x^4 - x^2 - 1$ | 1.570147 | 0.006992 | $x^4 - 2x^2 - 2x + 2$ |
| $x^5 - 2x^4 + x^3 - x^2$ $+x - 1$ | 1.673649 | 0.009705 | $x^4 - x^3 - x^2 - 2x + 3$ |
| $x^5 - x^4 - x^3 - 1$ | 1.704903 | 0.030844 | $2x^3 - 3x^3 - 2$ |
| $x^6 - x^5 - x^4 + x^2 - 1$ | 1.501595 | 0.0003491 | $x^5 + 2x^4 - 4x^3 - 3x^2$ $+3x - 2$ |
| $x^6 - 2x^5 + x - 1$ | 1.967168 | 0.032831 | $x - 2$ |
| $x^7 - x^5 - x^4 - x^3$ $-x^2 - x - 1$ | 1.590005 | 0.0001137 | $4x^6 - 5x^5 - x^4 - x^3$ $+x - 6$ |
| $x^7 - x^6 - x^4 - x^2 - 1$ | 1.601347 | 0.0004642 | $2x^5 - x^4 - 3x^3 - 1x^2$ $-x + 2$ |
| $x^7 - 2x^6 + x^5 - x^4$ $+x^3 - x^2 + x - 1$ | 1.640728 | 0.0003030 | $2x^6 - 2x^5 - 2x^4$ $-2x^2 + x + 3$ |
| $x^7 - 2x^6 + x^5 - 2x^4$ $+2x^3 - x^2 + x - 1$ | 1.790223 | 0.0006021 | $x^6 - 3x^5 + 5x^4 - 4x^3$ $-4x + 1$ |
| $x^7 - 2x^6 + x - 1$ | 1.983861 | 0.016138 | $x - 2$ |
| $x^9 - x^8 - x^7 - x^6$ $+x^4 + x^3 - 1$ | 1.743836 | 0.000004806 | $x^8 - 3x^5 - 4x^4 - 3x^2$ $+4x + 2$ |

Table 1: Pisot numbers where $l(q) = a(q)$.

| Pisot polynomial | Pisot number |
|---|---|
| $x^6 - x^5 - 2x^4 + x^2 - x - 1$ | 1.979476326 |
| $x^6 - 3x^5 + 3x^4 - 2x^3 + x - 1$ | 1.955451068 |
| $x^8 - x^7 - x^6 - x^5 - 2x^4 + 1$ | 1.995777793 |
| $x^9 - x^8 - x^7 - 2x^6 + x^3 - x^2 - x - 1$ | 1.997784254 |
| $x^9 - 2x^8 + x^5 - 2x^4 + x - 1$ | 1.996283920 |
| $x^9 - x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - 1$ | 1.994016415 |
| $x^9 - 2x^7 - 3x^6 - 2x^5 + x^3 - x - 1$ | 1.992483962 |
| $x^9 - x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x - 1$ | 1.989944545 |
| $x^9 - x^8 - x^7 - x^6 - x^5 - x^4 + 1$ | 1.963515789 |
| $x^{10} - x^9 - x^8 - x^7 - x^6 - 2x^5 + 1$ | 1.998987762 |
| $x^{10} - x^9 - 2x^8 + x^6 - x^5 - 2x^4 + x^2 - x - 1$ | 1.998772685 |
| $x^{10} - 2x^9 + x^7 - 2x^6 + x^4 - 2x^3 + x - 1$ | 1.998277927 |
| $x^{10} - 2x^9 + x^5 - x^4 - x + 1$ | 1.969456013 |
| $x^{10} - x^9 - 2x^8 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1$ | 1.966884957 |
| $x^{10} - x^9 - x^8 - x^7 - x^6 - x^5 + 1$ | 1.964715641 |

Table 2: Pisot polynomials that do not divide a $\pm 1$ polynomial.

Precise values of $l(q)$ and $a(q)$ at $q$ can be found on the web at

http://www.cecm.sfu.ca/~kghare, 2000.

| Pisot polynomial | $q$ | $l(q)$ | Approximate size of spectrum in $[\alpha_l, \alpha_u]$ | CPU secs |
|---|---|---|---|---|
| $x^{10} - x^9 - x^8 - x^7 + x^6 - x^3 + 1$ | 1.742975573 | 1.18668e-07 | 26973910 | 39m50s |
| $x^{10} - x^9 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - 1$ | 1.746541923 | 7.04603e-08 | 41498130 | 58m41s |
| $x^{10} - x^9 - x^8 - x^7 + x^5 - x^3 + 1$ | 1.795572823 | 3.5123e-08 | 43357472 | 1h1m7s |
| $x^{10} - x^9 - x^8 - x^7 - x^3 + 1$ | 1.852234868 | 8.17922e-08 | 25981420 | 34m38s |
| $x^{10} - x^9 - x^8 - x^7 - x^5 + x^4 + 1$ | 1.860952864 | 3.80874e-07 | 24944436 | 35m22s |
| $x^{10} - 2x^9 + x^8 - 2x^7 + x^6 + x^3 - x^2 + x - 1$ | 1.870250440 | 4.44816e-08 | 46252634 | 1h4m56s |
| $x^{10} - 2x^9 + x^7 - x^6 - x^3 + x^2 - 1$ | 1.881601063 | 2.57611e-07 | 27513576 | 35m35s |
| $x^{10} - 2x^8 - 3x^7 - x^6 + x^5 + 2x^4 + x^3 - x^2 - 2x - 1$ | 1.890027098 | 2.67873e-07 | 20923016 | 29m43s |
| $x^{10} - 2x^9 + x^8 - x^7 - x^6 - x^2 + x - 1$ | 1.903832902 | 2.22525e-07 | 22738454 | 28m42s |
| $x^{10} - x^9 - x^8 - x^7 - x^5 - x^4 - x^2 - x - 1$ | 1.921407084 | 3.12296e-08 | 41511868 | 57m5s |
| $x^{10} - 2x^9 + x^8 - 2x^7 + x^6 - x^5 - x^2 - 1$ | 1.957362809 | 2.22214e-07 | 22336604 | 29m7s |
| $x^{10} - 2x^9 + x^7 - 2x^6 + x^4 - 2x^3 + x - 1$ | 1.998277927 | 2.447e-08 | 46943484 | 1h3m54s |

Table 3: Successful calculations with a spectrum over 20 million for $l(q)$.

| Pisot polynomial | $q$ | $a(q)$ | Approximate size of spectrum in $[\alpha_l, \alpha_u]$ | CPU secs |
|---|---|---|---|---|
| $x^{10} - x^9 - x^8 + x^2 - 1$ | 1.601755862 | 1.59445e-07 | 33921896 | 30m38s |
| $x^{10} - x^9 - x^8 - x^2 + 1$ | 1.632690733 | 1.03354e-07 | 21835702 | 17m30s |
| $x^{10} - 2x^9 + x^8 - x^7$ $+x^3 - x^2 + x - 1$ | 1.735143707 | 8.28149e-08 | 32342934 | 29m18s |

Table 4: Successful calculations with a spectrum over 20 million for $a(q)$.

## Some questions of $A(q)$ for non-Pisot numbers $q$.

Peres and Solomyak asked for which $q$ in $1 < q < 2$ is $A(q)$ dense. It was unknown to the authors then if there were any $q$ for which $q$ is not a Pisot number and $A(q)$ is not dense. This we answered in the affirmative.

The examples of non-Pisot numbers $q$ where $A(q)$ is discrete required a search of 1868 possible candidates. The search rests on the following theorem.

**Theorem 1.** *If $q$ does not satisfy a polynomial of the form $\epsilon_n x^n + .. + \epsilon_m x^m + \beta_{m-1} x^{m-1} + \cdots + \beta_0$ where $\epsilon_i \in \{\pm 1\}$ and $\beta_i \in \{\pm 2, 0\}$, then $A(q)$ is not discrete.*

**Corollary 1.** *If $q$ does not satisfy a height 2 polynomial, then $A(q)$ is not discrete.*

**Observations on A(q).**

1. All examples of $q$ where $A(q)$ is discrete found are Perron numbers (all conjugates are of modulus less than $q$).
2. There were 120 examples found of non-Pisot numbers with discrete spectra $A(q)$.
3. There were 7 Salem numbers found (all but one conjugate is of modulus 1), with discrete spectra.
4. The only non-Pisot numbers $q$ whose minimal polynomial has Mahler measure less than 2 while $A(q)$ is discrete seems to be these Salem numbers.
5. The smallest (non-Pisot) number found with discrete spectrum is the Salem number 1.72208 of degree four (the root of $x^4 - x^3 - x^2 - x + 1$).
6. ** The largest root of $x^n - x^{n-1} - \cdots - x + 1$ is a Salem number with discrete spectrum and and the only Salem numbers of degree 9 or less with discrete spectrum satisfy a polynomial of this type.
7. The smallest degree polynomial satisfying a $q$ such that $A(q)$ is discrete is $x^3 - 2x - 2$.

| Non-Pisot polynomial | Root |
|---|---|
| $x^3 - 2x - 2$ | 1.769292354 |
| $x^4 - x^3 - 2x - 2$ | 1.873708564 |
| $x^4 - 2x^2 - 2x - 2$ | 1.899321089 |
| $x^5 - x^4 - 2x^2 - 2$ | 1.803707279 |
| $x^5 - x^4 - x^3 - 2x^2 + 2$ | 1.917514202 |
| $x^5 - x^4 - 2x^2 - 2x - 2$ | 1.942887561 |
| $x^5 - 2x^3 - 2x^2 - 2x - 2$ | 1.953501637 |
| $x^6 - 2x^4 - 2x^3 - 2$ | 1.813277575 |
| $x^6 - x^5 - x^4 - 2x^3 + 2x + 2$ | 1.859080768 |
| $x^6 - 2x^4 - 2x^3 - 2x^2 + 2$ | 1.865843123 |
| $x^6 - x^5 - x^4 - 2x^3 + 2$ | 1.961038629 |
| $x^6 - 2x^4 - 2x^3 - 2x^2 - 2x - 2$ | 1.977807115 |
| $x^6 - x^5 - x^4 - x^3 - 2x^2 + 2$ | 1.963984556 |
| $x^7 - x^6 - x^5 - x^4 + x^3 - 2x^2 + 2$ | 1.815396315 |
| $x^7 - x^6 - x^5 - 2x^4 + 2x^2 + 2$ | 1.888840344 |
| $x^7 - x^6 - x^5 - x^4 - x^3 + 2$ | 1.903972308 |
| $x^7 - x^6 - x^5 - 2x^4 + 2x + 2$ | 1.937730036 |
| $x^7 - x^6 - x^5 - x^4 - x^3 - 2x^2 + 2x + 2$ | 1.945197233 |
| $x^7 - x^6 - x^5 - 2x^4 + 2$ | 1.981204104 |
| $x^7 - x^6 - x^5 - x^4 - 2x^3 + 2$ | 1.982546502 |
| $x^7 - x^6 - x^5 - x^4 - x^3 - 2x^2 + 2$ | 1.983151826 |

Table 5: Polynomials with non-uniformly discrete spectrum

**Finding $\pm 1$ polynomials with LLL.**

It is of interest to see which $q$ satisfy a $\pm 1$ polynomial, or equivalently when $0 \in A(q)$. (The preceding Algorithm works only when the spectrum is discrete.)

For this we consider the following LLL based algorithm:

**Lemma 3.** *If the head coefficient and tail coefficient of $p(x)$ are both odd, then $p(x) | 1 + x + \cdots x^n \pmod 2$ for some $n$.*

Our algorithn will first, for input $p(x)$, find a polynomial $q(x)$, such that $p(x)q(x)$ is of the form $1 + x + \cdots + x^n \pmod 2$. (This is easy.)

From here LLL can often be used to find a polynomial $q'(x)$ where $q'(x)p(x)$ is a polynomial with $\pm 1$ coefficients.

For a basis choose $p(x)q(x)$ as one basis element, and $2p(x)x^n$ for $0 \le n < \deg q$ as the rest. It can be seen that if there is one basis element in the original basis with all odd coefficient, and all the rest have only even coefficients, then the resulting basis from LLL will have at least one basis element with only odd coefficients.

LLL tries to minimize the sum of the squares of the coefficients. The basis polynomial in the reduced basis with odd coefficients will, with some luck, be a polynomial with $\pm 1$ coefficients.

This works quite well for small problems. Unfortunately, for almost every polynomial of degree $n$, the resulting $q(x)$ is of degree $2^n - n$. This means that LLL must be performed on a basis of size $2^n - n$, which leads to an exponential time algorithm.

| Pisot polynomial | $\pm 1$ polynomial it divides |
|---|---|
| $x^2 - x - 1$ | $x^2 - x - 1$ |
| $x^3 - x - 1$ | $x^6 + x^5 - x^4 - x^3 - x^2 - x - 1$ |
| $x^3 - x^2 - 1$ | $x^6 - x^5 + x^4 - x^3 - x^2 - x - 1$ |
| $x^5 - x^3 - x^2 - x - 1$ | $-x^{30} + x^{29} + x^{28} - x^{27} + x^{26} + x^{25} - x^{24} - x^{23}$ $+x^{22} - x^{21} + x^{20} + x^{19} - x^{18} - x^{17} + x^{16} + x^{15}$ $+x^{14} - x^{13} - x^{12} + x^{11} - x^{10} + x^9 + x^8 - x^7 - x^6$ $+x^5 - x^4 - x^3 - x^2 - x - 1$ |
| $x^5 - x^4 - x^3 - x^2 + 1$ | $-x^{30} + x^{29} + x^{28} + x^{27} - x^{26} + x^{25} - x^{24} + x^{23}$ $+x^{22} - x^{21} - x^{20} + x^{19} - x^{18} - x^{17} - x^{16} + x^{15}$ $+x^{14} - x^{13} + x^{12} + x^{11} + x^{10} - x^9 - x^8 - x^7 - x^6$ $-x^5 - x^4 - x^3 - x^2 + x + 1$ |
| **Salem polynomial** | **$\pm 1$ polynomial it divides** |
| $x^{10} + x^9 - x^7 - x^6 - x^5$ $-x^4 - x^3 + x + 1$ | $-x^{30} - x^{29} + x^{28} + x^{27} + x^{26} + x^{25} - x^{24} + x^{23}$ $-x^{22} - x^{21} - x^{20} + x^{19} + x^{18} + x^{17} - x^{16} - x^{15}$ $+x^{14} + x^{13} - x^{12} + x^{11} - x^{10} - x^9 - x^8 - x^7 + x^6$ $+x^5 + x^4 + x^3 + x^2 - x - 1$ |
| $x^{10} - x^6 - x^5 - x^4 + 1$ | $-x^{30} - x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} - x^{23}$ $-x^{22} - x^{21} - x^{20} + x^{19} + x^{18} + x^{17} + x^{16} - x^{15}$ $-x^{14} - x^{13} - x^{12} + x^{11} - x^{10} + x^9 + x^8 - x^7 + x^6$ $-x^5 + x^4 + x^3 - x^2 + x - 1$ |
| $x^{10} - x^7 - x^5 - x^3 + 1$ | Degree 32 example found. |
| **Non-Pisot Non-Salem polynomial** | **$\pm 1$ polynomial it divides** |
| $x^4 - x^2 - 1$ | $x^5 + x^4 - x^3 - x^2 - x - 1$ |
| $x^6 - x^4 - 1$ | $x^{13} + x^{12} - x^{11} - x^{10} + x^9 + x^8$ $-x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ |
| $x^6 - x^4 - x^2 - 1$ | $x^7 + x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ |
| $x^4 + x^3 - x^2 - x - 1$ | $x^4 + x^3 - x^2 - x - 1$ |
| $x^4 - x - 1$ | $x^{14} + x^{13} + x^{12} - x^{11} - x^{10}$ $-x^9 - x^8 - x^7 - x^6 - x^5 + x^4 - x^3 - x^2 - x - 1$ |

Table 6: Polynomials found by the LLL algorithm

**Open Questions:**

1. Is the set of $q$ such that $a(q) = l(q)$ finite?
2. Does there exists and $\alpha \approx 1.95$ such that if $q < \alpha$, and $q$ Pisot, then $0 \in A(q)$?
3. Are all $q$ where $A(q)$ is discrete necessarily Perron?
4. Are the only $q$ where $A(q)$ is discrete and the Mahler measure of $q$ is less than 2 necessarily Salem numbers or Pisot numbers.
5. Do the only Salem numbers $q$ with $A(q)$ discrete satisfy $q^n - q^{n-1} - \cdots - q + 1$ for some $n$?
6. Is there a $\alpha \approx 1.72$ such that if $q < \alpha$ and $q$ is not Pisot then $A(q)$ is not discrete?
7. Is it true that $l(q) > 0$ if and only if $q$ is Pisot?