VIGNETTES INVOLVING RATIONAL FUNCTIONS, EXPONENTIAL SUMS OR ORTHOGONALITY.

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Müntz's Theorem and Friends.

A very attractive variant of Weierstrass' theorem characterizes exactly when the linear span of a system of monomials

$$\mathcal{M} := \{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $C[0,1]$ or $L_2[0,1]$.

Müntz's Theorem in $C[0,1]$. *Suppose* $\{\lambda_i\}_{i=1}^{\infty}$ *is a sequence of distinct positive real numbers not converging to* 0. *Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *in the uniform norm if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

This theorem follows by a simple trick from the L_2 version of the theorem.

Müntz's Theorem in $L_2[0,1]$. *Suppose* $\{\lambda_i\}_{i=0}^{\infty} \subset (-1/2, \infty)$ *is a sequence of distinct real numbers not converging to* $-1/2$. *Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_2[0,1]$ *if and only if*

$$\sum_{\substack{i=0\\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty.$$

(Full Müntz in L_p). *Let* $p \in [1, \infty]$. *Suppose* $\{\lambda_i\}_{i=0}^{\infty}$ *is a sequence of distinct real numbers greater than* $-1/p$. *Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_p[0,1]$ *if and only if*

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \infty.$$

Full Müntz in $C[0,1]$. (B&E). *Suppose* $\{\lambda_i\}_{i=1}^{\infty}$ *is a sequence of distinct, positive real numbers. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ *if and only if*

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2+1} = \infty.$$

Orthonormal Müntz-Legendre polynomials.

We can orthonormalize

$$\operatorname{span}\{x^{\lambda_0}, \dots, x^{\lambda_n}\}$$

Define L_n^*, the n-th orthonormal Müntz-Legendre polynomial defined by

$$L_n(x) := \frac{1}{2\pi i} \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \overline{\lambda}_k + 1}{t - \lambda_k} \frac{x^t dt}{t - \lambda_n}$$

$$= \sum_{k=0}^{n} c_{k,n} x^{\lambda_k}, \qquad x \in (0, \infty)$$

with

$$c_{k,n} := \frac{\prod_{j=0}^{n-1} (\lambda_k + \overline{\lambda}_j + 1)}{\prod_{j=0, j \neq k}^{n} (\lambda_k - \lambda_j)}$$

and

$$L_n^* := (1 + \lambda_n + \overline{\lambda}_n)^{1/2} L_n.$$

Then we get an orthonormal system, that is,

$$\int_0^1 L_n^*(x) \overline{L_m^*(x)} dx = \delta_{m,n}, \quad m, n = 0, 1, \dots .$$

Proof of Müntz's Theorem. We consider the approximation to x^m by elements of

$$\text{span}\{x^{\lambda_0}, \ldots, x^{\lambda_{n-1}}\}$$

in $L_2[0,1]$

For L_n^*, the n-th orthonormal Müntz-Legendre polynomial we have

$$L_n^*(x) = \sum_{i=0}^{n-1} a_i x^{\lambda_i} + a_n x^m$$

where

$$|a_n| = \sqrt{1+2m} \prod_{i=0}^{n-1} \left| \frac{m + \lambda_i + 1}{m - \lambda_i} \right|.$$

It follows from $\|L_n^*\|_{L^2[0,1]} = 1$ and orthogonality that L_n^*/a_n is the error term in the best $L_2[0,1]$ approximation to x^m from

$$\text{span}\{x^{\lambda_0}, \ldots, x^{\lambda_{n-1}}\}.$$

Therefore

$$\min_{b_i \in \mathbb{C}} \left\| x^m - \sum_{i=0}^{n-1} b_i x^{\lambda_i} \right\|_{L^2[0,1]}$$

$$= \frac{1}{|a_n|} = \frac{1}{\sqrt{1+2m}} \prod_{i=0}^{n-1} \left| \frac{m-\lambda_i}{m+\lambda_i+1} \right| .$$

So, for $m \neq \lambda_i$,

$$x^m \in \overline{\text{span}}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

(where $\overline{\text{span}}$ denotes the $L_2[0,1]$ closure of the span) if and only if

$$\limsup_n \prod_{i=0}^{n-1} \left| 1 - \frac{2m+1}{m+\lambda_i+1} \right| = 0.$$

□

Müntz Rationals.

A surprising and beautiful theorem, conjectured by Newman and proved by Somorjai , states that rational functions derived from any infinite Müntz system are always dense in $C[a,b], a \geq 0$. More specifically we have

Denseness of Müntz Rationals. *Let* $\{\lambda_i\}_{i=0}^{\infty}$ *be any sequence of distinct real numbers. Suppose* $a \geq 0$*. Then*

$$\left\{ \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\lambda_i}} : a_i, b_i \in \mathbb{R}, \quad n \in \mathbb{N} \right\}$$

is dense in $C[a,b]$*.*

The proof of this theorem, primarily due to Somorjai, rests on the existence of zoomers. A function Z defined on $[a,b]$ is called an ϵ-*zoomer* ($\epsilon > 0$) at $\zeta \in (a,b)$ if

$$\begin{aligned} Z(x) &> 0, & x &\in [a,b] \\ Z(x) &\leq \epsilon, & x &< \zeta - \epsilon \\ Z(x) &\geq \epsilon^{-1}, & x &> \zeta + \epsilon \end{aligned}$$

While (approximate) δ-functions are approximate building blocks for polynomial approximations, the existence of ϵ-zoomers is all that is needed for rational approximations.

A comparison between Müntz's Theorem and this shows the power of a single division in these approximations. In what other contexts does allowing a division create a spectacularly different result.

Conjecture 1 (Newman 1978). *If $\mathcal{M}$ is any infinite Markov system on $[0,1]$ then the set of rationals*

$$\left\{\frac{p}{q} : p, q \in \mathit{span}\ \mathcal{M}\right\}$$

is dense in $C[0,1]$.

He calls this a "wild conjecture in search of a counterexample". It does however hold for both

$$\mathcal{M} = \{x^{\lambda_0}, x^{\lambda_1}, \dots\}, \qquad \lambda_i \geq 0$$

and

$$\mathcal{M} = \left\{ \frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2} \cdots \right\}$$

We give a counterexample to this conjecture. However, the characterization of the class of Markov systems for which it holds remains as an interesting question.

Newman also conjectures the non-denseness of products

Conjecture 2 (Newman 1978).

$$\{\Sigma a_i x^{i^2}\}\{\Sigma b_i x^{i^2}\}$$

is not dense in $C[0,1]$.

He speculates that this "extra" multiplication of Müntz polynomials should not carry the utility of the "extra" division.

We will show that products of two Müntz polynomials from non-dense Müntz spaces never form a dense set in $C[0,1]$.

Non-Dense Ratios of Müntz Spaces. Suppose $0 \le \lambda_0 < \lambda_1 < \cdots$. Let $a > 0$. Show that

$$\left\{ \frac{\sum_{i=0}^n a_i x^{\lambda_i}}{\sum_{i=0}^n b_i x^{-\lambda_i}} : a_i, b_i \in \mathbb{R},\ n \in \mathbb{N} \right\}$$

is dense in $C[a,b]$, if and only if $\sum_{i=1}^\infty 1/\lambda_i < \infty$.

The following example is due to Boris Shekhtman and P. B.

A Markov System with Non-Dense Rationals.

We construct an infinite Markov system as follows. Consider non-negative even integers

$$0 = \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 <$$

$$\cdots < \lambda_n < \mu_n < \cdots$$

which are lacunary in the sense that there exists $q > 1$ so that

$$\frac{\mu_i}{\lambda_i} > q \quad \text{and} \quad \frac{\lambda_{i+1}}{\mu_i} > q, \qquad i = 1, 2, \ldots .$$

Let $\varphi_k \in C[-1, 1]$ be defined by

$$\varphi_0 := 1, \quad \varphi_{2k}(x) := x^{\mu_k}$$

and

$$\varphi_{2k+1}(x) := \begin{cases} x^{\lambda_k}, & x \geq 0 \\ -x^{\lambda_k}, & x \leq 0. \end{cases}$$

Then $\{\varphi_0, \varphi_1, \ldots\}$ is a Markov system on $[-1, 1]$.

But the rational functions of the form

$$\frac{\sum_{j=0}^{n} a_j \varphi_j}{\sum_{j=0}^{m} b_j \varphi_j}, \quad a_j, b_j \in \mathbb{R}, \quad n, m \in \mathbb{N}$$

are not dense in $C[-1, 1]$ in the uniform norm.

Revised Newman Conjecture. *If $\mathcal{M}$ is any infinite Descartes system on $[0,1]$ then the set of rationals*

$$\left\{ \frac{p}{q} : p, q \in \mathit{span}\ \mathcal{M} \right\}$$

is dense in $C[0,1]$.

Chebyshev Polynomials in Chebyshev Spaces.

Suppose

$$H_n := \operatorname{span}\{f_0, \dots , f_n\}$$

is a Chebyshev space on $[a, b]$ and A is a compact subset of $[a, b]$ with at least $n + 1$ points. We can define the *generalized Chebyshev polynomial*

$$T_n := T_n\{f_0, \dots , f_n; A\}$$

for H_n on A by

$$T_n = c\left(f_n - \sum_{k=0}^{n-1} a_k f_k\right)$$

where the numbers $a_0, a_1, \dots , a_n \in \mathbb{R}$ are chosen to minimize

$$\left\| f_n - \sum_{k=0}^{n-1} a_k f_k \right\|_A$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that

$$\|T_n\|_A = 1$$

Denseness and Zeros of Chebyshev Polynomials.

For a sequence of Chebyshev polynomials T_n associated with a fixed Markov system on $[a, b]$ we have the *mesh* of T_n is defined by

$$M_n := M_n(T_n : [a, b]) := \max_{1 \le i \le n+1} |x_i - x_{i-1}|.$$

where x_i are the zeros of T_n.

Theorem (P.B.). *If $\mathcal{M} := \{1, f_1, f_2, \ldots\}$ is an infinite Markov system on $[a, b]$ with each $f_i \in C^1[a, b]$. Then span $\mathcal{M}$ is dense in $C[a, b]$ if and only if*

$$\lim_{n \to \infty} M_n = 0$$

where M_n is the mesh of the associated Chebyshev polynomials.

Corollary (Weierstrass' Theorem). *The polynomials are dense in $C[-1, 1]$.*

Denseness and Unbounded Bernstein Inequalities.

Definition (Unbounded Bernstein Inequality). *Let $\mathcal{A}$ be a subset of $C^1[a,b]$. We say that $\mathcal{A}$ has an everywhere unbounded Bernstein inequality if*

$$\sup\left\{\frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[a,b]}} : p \in \mathcal{A}, \quad p \neq 0\right\} = \infty$$

for every $[\alpha,\beta] \subset [a,b], \quad \alpha \neq \beta$.

Bernstein-Type Inequality for Chebyshev Spaces. *Let $\{1, f_1, \ldots, f_n\}$ be a Chebyshev system on $[a,b]$ such that each f_i is differentiable at $x_0 \in [a,b]$. Let*

$$T_n := T_n\{1, f_1, \ldots, f_n; [a,b]\}$$

be the associated Chebyshev polynomial. Then

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[a,b]}} \leq \frac{2}{1 - |T_n(x_0)|} |T_n'(x_0)|$$

for every $p_n \in \operatorname{span}\{1, f_1, \ldots, f_n\}$ provided $|T_n(x_0)| \neq 1$.

Characterization of Denseness by Unbounded Bernstein Inequality. *Suppose* $\mathcal{M} := \{f_0 := 1, f_1, \dots\}$ *is an infinite Markov system on* $[a, b]$ *with each* $f_i \in C^2[a, b]$ *. Then* span $\mathcal{M}$ *is dense in* $C[a, b]$ *if and only if* span $\mathcal{M}$ *has an everywhere unbounded Bernstein inequality.*

Corollary (Weierstrass' Theorem. *The polynomials are dense in* $C[-1, 1]$.

Classical Polynomial Inequalities.

Remez Inequality. *The inequality*

$$\|p\|_{[-1,1]} \leq T_n\left((2+s)/(2-s)\right)$$

holds for every $p \in \mathcal{P}_n$ *and* s *satisfying*

$$m\left(\{x \in [-1,1] : |p(x)| \leq 1\}\right) \geq 2-s.$$

Here T_n *is the Chebyshev polynomial:*

$$T_n(x) := \cos(n \arccos x).$$

Bernstein's Inequality. *For* $p \in \mathcal{P}_n^c$

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}}\|p\|_{[-1,1]}, \quad -1 < x < 1.$$

Markov's Inequality. *For* $p \in \mathcal{P}_n^c$

$$\|p'\|_{[-1,1]} \leq n^2\|p\|_{[-1,1]}.$$

• The L_p analogue of Markov's Inequality states that

$$\int_{-1}^{1} |Q'(x)|^p dx \le c^{p+1} n^{2p} \int_{-1}^{1} |Q(x)|^p dx$$

for every $Q \in \mathcal{P}_n$ and $0 < p < \infty$, where c is an absolute constant.

• We will prove this more generally with a constant 12.

• The best possible Markov factor in L_p is still an open problem even for $p = 2$ or $p = 1$.

Müntz Systems (Dirichlet Sums).

The system

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \quad \text{on } [0,1]$$

is called a *Müntz systems.*

Müntz-Chebyshev polynomials.

In principal it is possible to construct an analogue of the Chebyshev Polynomial for a Müntz System

$$\mathrm{span}\{x^{\lambda_0}, \ldots, x^{\lambda_n}\}$$

This will be an equioscillating "polynomial" and will be extremal for a number of problems

One needs these in the proof of the Full Müntz Theorem.

In particular one needs the characterization of denseness of an infinite Markov system in terms of denseness of the zeros of the associated Chebyshev polynomials.

Inequalities in Müntz Spaces.

We first present a simplified version of Newman's beautiful proof of a Markov-type inequality for Müntz polynomials. This modification allows us to prove the L_p analogues of Newman's Inequality.

Newman's Inequality. *Let* $\{\lambda_i\}_{i=0}^{\infty}$ *be a sequence of distinct nonnegative real numbers. Then*

$$\frac{\|xp'(x)\|_{[0,1]}}{\|p\|_{[0,1]}} \leq 9 \sum_{j=0}^{n} \lambda_j$$

for every p *in the linear span of*

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

In L_p *we must replace the constant 9 by 13 .*

For $p \geq 1$ and $P \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ with exponents λ_j greater than $-1/p$.

Sharp Markov Inequality. (B&E)

$$\|xP'(x)\|_{L_p[0,1]} \leq$$

$$13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right) \|P\|_{L_p[0,1]}$$

Nikolskii-type Inequality. (B&E)

$$\|y^{1/p} P(y)\|_{L_\infty[0,1]} \leq$$

$$13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right)^{1/p} \|P\|_{L_p[0,1]}$$

- Note the implication for Müntz's Theorem with exponents tending to $-1/p$.

Lorentz's Problem for Müntz Polys.

Conjecture.

$$\sup_p \frac{|\, p'(1)\, |}{||p||_{[0,2]}} \leq ??$$

where the sup is over all Müntz polynomials

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j \geq 0$$

independent of the exponents.

• These following results improve inequalities of Lorentz and Schmidt and others going back 25 years.

• Lorentz finally conjectured the above with a $C{*}n$ bound?

Theorem. *For every* $0 < a < b$

$$\sup_p \frac{\mid p'(a) \mid}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}$$

The sup is over all Müntz polynomials

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j \geq 0.$$

Theorem 3.2. *The inequality*

$$\sup_{0 \neq f} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}$$

holds for every $y \in (a,b)$. *where*

$$f(t) = a_0 + \sum_{i=1}^{n} a_j e^{\lambda_j t}, \;\; a_j, \lambda_j \in \mathbb{R}.$$

A Remez Inequality for Müntz Spaces.

• This Remez-type inequality allows us to resolve two reasonably long standing conjectures.

• The first, due to D. J. Newman and dating from 1978, asserts that if

$$\sum_{i=1}^{\infty} 1/\lambda_i < \infty$$

then the set of products

$$\{p_1 p_2 : p_1, p_2 \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}\}$$

is not dense in $C[0,1]$.

• The second is a complete extension of Müntz's classical theorem on the denseness of Müntz spaces in $C[0,1]$ to denseness in $C[A]$, where $A \subset [0,\infty)$ is an arbitrary compact set with positive Lebesgue measure.

Müntz's Theorem Generalized. *For an arbitrary compact set $A \subset [0, \infty)$ with positive Lebesgue measure,*

$$span\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \qquad \lambda_i \geq 1$$

is dense in $C[A]$ if and only if

$$\sum_{i=1}^{\infty} 1/\lambda_i = \infty.$$

• Let

$$p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ The most useful form of our Remez inequality states:

Bounded Remez Inequality. (B&E).

For every sequence $\{\lambda_i\}_{i=0}^{\infty}$ *satisfying*

$$\sum_{i=1}^{\infty} 1/\lambda_i < \infty$$

there is a constant c *depending only on* $\{\lambda_i\}_{i=0}^{\infty}$ *and* s *(and not on* n, ϱ, *or* A*) so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every Müntz polynomial p, *as above, associated with* $\{\lambda_i\}_{i=0}^{\infty}$, *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* $s > 0$.

Sharp Extensions of Bernstein's Inequality to Rational Spaces.

Let

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; A)$$

$$:= \left\{ \frac{p_n(z)}{\prod_{j=1}^{n}(z - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

where the A indicates that the poles are to avoid A.

• If the a_i tend to infinity we recover the ordinary polynomials. So the following results are sharp extensions of the usual Bernstein inequality.

•These are also sharp extensions of results of Russak and others.

Bernstein-Szegő Type Inequality. (B&E).

For $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$, *let*

$$B_n(x) := \sum_{k=1}^n \operatorname{Re}\left(\frac{\sqrt{a_k^2-1}}{a_k-x}\right)$$

where the root $\sqrt{a_k^2-1}$ *is determined by*

$$c_k := a_k - \sqrt{a_k^2-1}, \qquad |c_k| < 1.$$

Then

$$(1-x^2)f'(x)^2 + B_n(x)^2 f(x)^2 \le B_n(x)^2 \|f\|_{[-1,1]}^2$$

and

$$|f'(x)| \le \frac{1}{\sqrt{1-x^2}} B_n(x) \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$.

Theorem. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$. *Then*

$$|f'(z_0)|/\|f\|_{\partial D} \leq$$

$$\max\left\{ \sum_{\substack{j=1 \\ |a_j|>1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j|<1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$.

Theorem. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(x_0)|/\|f\|_{\mathbb{R}} \leq$$

$$\max\left\{ \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)>0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2}, \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)<0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2} \right\}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \mathbb{R})$.

Inequalities for p_n'/p_n.

These are metric inequalities of the form

$$m\left(\left\{x : \frac{r_n'(x)}{r_n(x)} \geq \alpha\right\}\right) \leq \frac{\beta \cdot n}{\alpha}, \qquad \alpha > 0$$

where r_n is a rational function of type (n, n) and β is a constant independent of n. Here m is Lebesgue measure.

Theorem (Loomis). *If $p_n \in \mathcal{P}_n$ has n real roots then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p_n'(x)}{p_n(x)} \geq \alpha\right\}\right) = \frac{n}{\alpha} \quad \textit{for } \alpha > 0$$

Proof. Draw a picture of

$$\frac{p_n'(x)}{p_n(x)}$$

Lemma. *If* $p_n \in \mathcal{P}_n$ *is positive on* $[a,b]$ *then there exists* q_n, $s_n \in \mathcal{P}_n$ *nonnegative on* $[a,b]$ *with all real roots (in* $[a,b]$*) so that* $p_n(x) = q_n(x) + s_n(x)$.

Theorem. *Let* $p_n \in \mathcal{P}_n$ *then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p_n'(x)}{p_n(x)} \geq \alpha\right\}\right) \leq \frac{2n}{\alpha}, \qquad \alpha > 0.$$

Theorem. *If* $r_n = p_n/q_n \in \mathcal{R}_{n,n}n$ *then*

$$m\left(\left\{x \in \mathbb{R} : \frac{r_n'(x)}{r_n(x)} \geq \alpha\right\}\right) \leq \frac{8n}{\alpha}, \qquad \alpha > 0.$$

• It would be interesting to know the right constant above. It might well be 2π This is closely related to the incomplete rational problem concerning the interval of denseness of

$$\{\exp(-nx)p_n(x)/q_n(x)\}.$$

Incomplete Rationals

We consider rational approximations of the form

$$\left\{ (1+z)^{\alpha n+1} \frac{p_{cn}(z)}{q_n(z)} \right\}$$

in certain natural regions in the complex plane where p_{cn} and q_n are polynomials of degree cn and n respectively.

In particular we construct natural maximal regions (as a function of α and c) where the collection of such rational functions is dense in the analytic functions.

So from this point of view we have rather complete analogue theorems to the results concerning incomplete polynomials on an interval.

The analysis depends on a careful examination of the zeros and poles of the Padé approximants to $(1+z)^{\alpha n+1}$. This is effected by an asymptotic analysis of certain integrals.

In this sense it mirrors the well known results of Saff and Varga on the zeros and poles of the Padé approximant to exp. Results that, in large measure, we recover as a limiting case.

In order to make the asymptotic analysis as painless as possible we prove a fairly general result on the behavior, in n, of integrals of the form

$$\int_0^1 [t(1-t)f_z(t)]^n dt$$

where $f_z(t)$ is analytic in z and a polynomial in t.

From this we can and do analyze automatically (by computer) the limit curves and regions that we need.

The Wellspring:

In a remarkable paper of 1924, Szegő considered the zeros of the partial sums $s_n(z) := \sum_{k=0}^{n} z^k/k!$ of the MacLaurin expansion for e^z. Szegő established that $\hat{z}$ is a limit point of zeros of the sequence of normalized partial sums,

$$\{s_n(nz)\}_{n=0}^{\infty},$$

if and only if

$$\hat{z} \in \{z : |ze^{1-z}| = 1, |z| \leq 1\}.$$

Moreover, Szegő showed that $\hat{z}$ is a nontrivial limit point of zeros of the normalized remainder

$$\{e^{nz} - s_n(nz)\}_{n=1}^{+\infty}$$

if and only if

$$\hat{z} \in \{z : |ze^{1-z}| = 1, |z| \geq 1\}.$$

Padé Approximation to $(1+z)^{\alpha n}$:

Theorem. *The set of functions*

$$\{(1+z)^{\alpha n} r_n(z) : r_n(z) \in \pi_{n,n}\}_{n=1}^{+\infty}$$

is dense in $A(K)$, the analytic functions on K, where K is an arbitrary compact subset of R_3 and not in any region strictly containing R_3 (R_3 is the region in Fig.1).

Theorem 2.1. *For the (m,n) Padé approximation to $(1+z)^{\alpha n+1}$ at 0, $\alpha > 0$, we have*

(a)
$$(1+z)^{\alpha n+1} - \frac{p_m(z)}{q_n(z)}$$
$$= \frac{z^{m+n+1} \int_0^1 (1-t)^m t^n (1+tz)^{\alpha n - m} dt}{q_n(z)},$$

(b) $$p_m(z) = \int_0^1 (t-1)^n t^{\alpha n - m} (1+z-t)^m dt,$$

and

(c) $$q_n(z) = \int_0^1 (1-t)^m t^{\alpha n - m} (t(z+1)-1)^n dt.$$

Corollary 2.2. *For the (cn, n) Padé approximation to $(1+z)^{\alpha n+1}$ at 0, $\alpha > 0$, we have*

(a) $$(1+z)^{\alpha n+1} - \frac{p_{cn}(z)}{q_n(z)}$$

$$= \frac{z^{cn+n+1} \int_0^1 [(1-t)^c t(1+tz)^{\alpha - c}]^n \, dt}{q_n(z)},$$

(b) $$p_{cn}(z) = \int_0^1 [(t-1)t^{\alpha - c}(1+z-t)^c]^n \, dt,$$

and

(c) $$q_n(z) = \int_0^1 [(1-t)^c t^{\alpha - c}(t(1+z)-1)]^n \, dt.$$

Corollary 2.3. *When $c = 1$, we have*

$$(1+z)^n p_n\left(\frac{-z}{1+z}\right) = q_n(z).$$

The Asymptotic Analysis

Theorem 2.4. *Let*

$$I_n = \int_0^1 [t(1-t)f(t)]^n dt = \int_0^1 [Q(t)]^n dt$$

where $Q(t) = t(1-t)f(t)$ *is a polynomial of degree* N *in* t.

Let $t_1, t_2, \ldots, t_{N-1}$ *be the* $N-1$ *zeros of* $Q'(t)$.

Suppose that

$$|Q(t_i)| \neq |Q(t_j)|, \qquad i \neq j.$$

Then

$$\lim_{n\to\infty} I_n^{1/n} = arg(Q(t_i))|Q(t_i)| = Q(t_i)$$

for some i.

Theorem 2.5. *Let*

$$I_n(z) = \int_0^1 [t(1-t)f_z(t)]^n dt = \int_0^1 [Q_z(t)]^n dt$$

where $Q_z(t) = t(1-t)f_z(t)$ *is a polynomial in* t *and analytic in* z *on an open connected set* U*. Suppose*

$$|Q_z(t_i(z))| \neq |Q_z(t_j(z))|$$

for any $i \neq j$*, and any* $z \in U$*, where* $t_i := t_i(z)$ *are the zeros of the polynomial* $\frac{d}{dt}Q_z(t)$ *(which by the above assumption can be given so that each* t_i *is analytic on* U*). Then*

(a) $I_n(z)^{1/n}$ *converges to a non-zero limit pointwise on* U*.*

(b) $|I_n(z)|^{1/n}$ *is uniformly bounded on compact subsets of* U*.*

(c) $I_n(z)^{1/n}$ *converges uniformly to a* $Q_z(t_i(z))$ *on compact subsets of* U*, and* $Q_z(t_i(z))$ *is analytic on* U*. Moreover,* $Q_z(t_i(z)) \neq 0$ *for all* $z \in U$ *.*

Corollary 2.6. *Let $I_n(z)$, $f_z(t)$ and $Q_z(t)$ be as in Theorem 2.5. Suppose that for each z, $Q_z(t)$ is a polynomial of degree N in t, and further that $Q_z(t)$ is analytic in z. Then, the limit points of the zeros of $I_n(z)$ can only cluster on the curve*

$$\{z : |Q_z(t_i(z))| = |Q_z(t_j(z))|, \qquad \textit{for some } i \neq j\}$$

or at points where $Q_z(t_i(z)) = 0$, or at points where $Q_z(t_i(z))$ is not analytic. (Note $t_i := t_i(z)$, which is a function of z.)

These results require some careful saddle point analysis.

The nice thing is that one can guarantee that the right contours exist without actually constructing them.

Computing the possible limit curves is now an exercise in computer algebra.

Specialization to $c = 1$

$$p_n(z) = \int_0^1 [(t-1)t^{\alpha-1}(1-t+z)]^n dt,$$
$$q_n(z) = \int_0^1 [(1-t)t^{\alpha-1}(t(1+z)-1)]^n dt,$$

$$e_{(\alpha-1)n}(z) = \int_0^1 [(1-t)t(1+tz)^{\alpha-1}]^n dt.$$

Let $Q_z(t) = (1-t)t^{\alpha-1}(t(1+z)-1)$, then

$$Q_z(0) = Q_z(1) = Q_z\left(\frac{1}{1+z}\right) = 0,$$

and

$$\frac{d}{dt} Q_z(t) \Big|_{t=t_{1,2}(z)} = 0$$

where

$$t_{1,2}(z) = \frac{\alpha(z+2) \pm \mu}{2(z+1)(1+\alpha)},$$

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

Therefore, from Corollary 2.6 and the above observation, the critical curve for $p_n(z)$, $q_n(z)$ and $e_{(\alpha-1)n}(z)$ is

$$\{z : |Q_z(t_1(z))| = |Q_z(t_2(z))|\},$$

which is

$$\left\{ \left| \frac{\alpha z + 2z + 2 + \mu}{\alpha z + 2z + 2 - \mu} \right| \left| \frac{\alpha z - 2 - \mu}{\alpha z - 2 + \mu} \right| \left| \frac{\alpha z + 2\alpha - \mu}{\alpha z + 2\alpha + \mu} \right|^{\alpha - 1} = 1 \right\}$$

where

$$\mu = (\alpha^2 z^2 + 4z + 4)^{1/2}.$$

The critical curves for $\alpha = 2$, $\alpha = 3$, $\alpha = 5$ and $\alpha = 8$ are shown in Figures 1, 2, 3 and 4 respectively.

In Figures 5 and 6, we plot the zeros of $p_n(z)$ and $q_n(z)$ for $\alpha = 2$, $n = 20$ and $\alpha = 3$, $n = 10$ respectively. We also plot the zeros of $e_{(\alpha-1)n}(z)$ for $\alpha = 3$, $n = 15$ in Figure 7.

Theorem 3.1. *For $\alpha > 1$, $\{q_n(z)\}^{1/n}$ converges to $Q_z(t_2(z))$ uniformly on any compact subset of R_1, R_2 and R_3, and to $Q_z(t_1(z))$ uniformly on any compact subset of R_4. Moreover, the limit points of the zeros of $\{q_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_3, which is the boundary between R_3 and R_4.*

Theorem 3.2. *For $\alpha > 1$, $\{p_n(z)\}^{1/n}$ converges to $(1+z)^{\alpha} Q_z(t_1(z))$ uniformly on any compact subset of R_1 and R_2, and to $(1+z)^{\alpha} Q_z(t_2(z))$ uniformly on any compact subset of R_3 and R_4. Moreover, the limit points of the zeros of $\{p_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_2, which is the boundary between R_2 and R_3.*

Theorem 3.3. *For $\alpha > 1$, $\{e_{(\alpha-1)n}\}^{1/n}$ converges to $(1+z)^{\alpha} Q_z(t_2(z))/z^2$ uniformly on any compact subset of R_1 and to $(1+z)^{\alpha} \varphi_z(t_1(z))/z^2$ uniformly on any compact subset of R_2, R_3 and R_4. Moreover, the limit points of the zeros of $\{e_n(z)\}_{n=1}^{\infty}$ are dense on the branch B_1, which is the boundary between R_1 and R_2.*

Theorem 4.1. *Let* $p_n(z)$, $q_n(z)$ *and* $e_{(\alpha-1)n}(z)$ *be as in Corollary 2.2 in the case* $c = 1$. *Then we have that* $(1+z)^{\alpha n+1} q_n(z)/p_n(z)$ *converges*

(a) *to* ∞ *uniformly on any compact subset of* R_1 *and* R_4 *(as in Figures 1,2,3,4);*

(b) *to* 0 *uniformly on any compact subset of* R_2*;*

(c) *to* 1 *uniformly on any compact subset of* R_3.

Remark. *Observe that* 1 *can not be approximated on any region strictly larger than* R_3 *by the Rouché's Theorem, so* R_3 *is a natural maximal region of denseness.*

Theorem 4.2.

$$\{(1+z)^{\alpha n} r_n(z) : r_n(z) \in \pi_{n,n}\}_{n=1}^{\infty}$$

is dense in $A(K)$ *where* K *is an arbitrary compact subset of* R_3.

The Polynomial Case: c=0

Theorem 5.1. *If c:=0 then* $q_n(z)$ *and* $e_{\alpha n}(z)$ *have the same critical curve*

$$\{z : |z(1+z)^{\alpha}| = \alpha^{\alpha}/(1+\alpha)^{1+\alpha}\}.$$

and the limit points of the zeros of $q_n(z)$ *or* $e_{\alpha n}(z)$ *can only cluster on this curve .*

Theorem 5.4.

$$\{(1+z)^{\alpha n} p_n(z) : p_n(z) \in \pi_n\}_{n=1}^{+\infty}$$

is dense in $A(K)$ *where* K *is an arbitrary compact subset of* R_3 *as in Fig. 8.*

The limit points of the zeros of $\{q_n(z)\}_{n=1}^{\infty}$ *are dense in the boundary between* R_1 *and* R_3*.*

The limit points of the zeros of $\{e_{\alpha n}(z)\}_{n=1}^{\infty}$ *are dense in the boundary between* R_1 *and* R_2*.*

Question. For which sets K is

$$\{(1+z)^{\alpha n} r_n(z)\}_{n=1}^{\infty}$$

dense in $A(K)$?

Uniform convergence to 1 of $\{x^{\theta n} r_n(x)\}$ is not possible on any interval $[b, 1]$ with

$$b < \tan^4(\pi(\theta - 1)/4\theta)$$

and this is essentially sharp. (Saff and Rachmanov)

Uniform convergence to 1 of $\{e^x r_n(x)\}$ is not possible on any interval $[0, a]$ with $a > 2\pi$ (compare $b = 2$ for polynomials).

Question. What is the maximum measure of

$$\{z : |r_n'(z)/r_n(z)| >= n\}$$

in the complex plane? (On the line it is 2π)