Preface

Polynomials pervade mathematics, and much that is beautiful in mathematics is related to polynomials. Virtually every branch of mathematics, from algebraic number theory and algebraic geometry to applied analysis, Fourier analysis, and computer science, has its corpus of theory arising from the study of polynomials. Historically, questions relating to polynomials, for example, the solution of polynomial equations, gave rise to some of the most important problems of the day. The subject is now much too large to attempt an encyclopedic coverage.

The body of material we choose to explore concerns primarily polynomials as they arise in analysis, and the techniques of the book are primarily analytic. While the connecting thread is the polynomial, this is an analysis book. The polynomials and rational functions we are concerned with are almost exclusively of a single variable.

We assume at most a senior undergraduate familiarity with real and complex analysis (indeed in most places much less is required). However, the material is often tersely presented, with much mathematics explored in the exercises, some of which are quite hard, many of which are supplied with copious hints, some with complete proofs. Well over half the material in the book is presented in the exercises. The reader is encouraged to at least browse through these. We have been much influenced by Pólya and Szegő's classic "Problems and Theorems in Analysis" in our approach to the exercises. (Though unlike Pólya and Szegő we chose to incorporate the hints with the exercises.)

The book is mostly self-contained. The text, without the exercises, provides an introduction to the material, but much of the richness is reserved for the exercises. We have attempted to highlight the parts of the theory and the techniques we find most attractive. So, for example, Müntz's lovely characterization of when the span of a set of monomials is dense is explored in some detail. This result epitomizes the best of the subject: an attractive and nontrivial result with several attractive and nontrivial proofs.

There are excellent books on orthogonal polynomials, Chebyshev polynomials, Chebyshev systems, and the geometry of polynomials, to name but a few of the topics we cover, and it is not our intent to rewrite any of these. Of necessity and taste, some of this material is presented, and we have attempted to provide some access to these bodies of mathematics. Much of the material in the later chapters is recent and cannot be found in book form elsewhere.

Students who wish to study from this book are encouraged to sample widely from the exercises. This is definitely "hands on" material. There is too much material for a single semester graduate course, though such a course may be based on Sections 1.1 through 5.1, plus a selection from later sections and appendices. Most of the material after Section 5.1 may be read independently.

Not all objects labeled with "E" are exercises. Some are examples. Sometimes no question is asked because none is intended. Occasionally exercises include a statement like, "for a proof see ... "; this is usually an indication that the reader is not expected to provide a proof.

Some of the exercises are long because they present a body of material. Examples of this include E.11 of Section 2.1 on the transfinite diameter of a set and E.11 of Section 2.3 on the solvability of the moment problem. Some of the exercises are quite technical. Some of the technical exercises, like E.4 of Section 2.4, are included, in detail, because they present results that are hard to access elsewhere.

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Contents