Research Problems

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Some Old and New Problems in Approximation Theory: Research Problems 95–1

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Problem I. Faber proved: let $-1 \le x_1 < x_2 < \ldots < x_n \le 1$, $\left(x_i = x_i^{(n)}\right)$, be an arbitrary sequence of n points. Then there are always values $|y_i| \le 1$ for which for the polynomial $P_{n-1}(x)$ of degree at most $n-1$ satisfying $P_{n-1}(x_i) = y_i$ we have

$$\max_{-1\le x\le 1} |P_{n-1}(x)| > c \log n,$$

and here $c > 0$ is an absolute constant. In fact, the main point is that $\max |P_n(x)| \to \infty$ as $n \to \infty$.

Zygmund and Bernstein considered the problem: are there points $-1 \le x_1 < \ldots < x_n \le 1$ such that for any choice of the $|y_i| = 1$ there is a polynomial of degree $< n(1+\ell_1)$ for which

$$P_n(x_i) = y_i \quad \text{and} \quad \max_{-1\le x\le 1} |P_n(x)| < \ell_2. \tag{1}$$

They proved that the roots of the Chebyshev and Legendre polynomials have this property. In my paper (Journal d'Analyse Math., **19**(1967), 135–148) I gave a necessary and sufficient condition for $x_1, x_2, \ldots, x_n$, which implies (1).

I think if the degree of $P_n(x)$ is $< n(1+o(1))$, i.e., the degree of $P_n(x)$ is $< n + f(n)$ where $f(n)/n \to 0$ as slowly as we wish, then (1) can never hold. This, I think, would be a significant extension of Faber's theorem.

Problem II. Let $\{x_i^{(n)}\}$, $1 \le i \le n$, $1 \le n < \infty$ be a point group, $-1 < x_i^{(n)} < 1$. Vértesi and I proved (Acta Math. Hung., **36**(1980), 71–89 and **38**(1981), 263) that there always is a continuous function $f(x)$, $-1 \le x \le 1$ for which the Lagrange interpolation polynomials $\mathcal{L}_n(f(x))$ taken at the point group $\{x_i^{(n)}\}$ diverge almost everywhere. It is easy to see that this is best possible, and that for a general point group divergence everywhere is not possible.

By an old theorem of Hahn divergence is possible at a point x_0 if and only if

$$\overline{\lim_{n\to\infty}} \sum_{k=1}^{n} \left|\ell_k^{(n)}(x_0)\right| = \infty \tag{2}$$

where $\ell_k^{(n)}(x)$ are the fundamental polynomials of Lagrange interpolation:

$$\ell_k^{(n)}\left(x_k^{(n)}\right) = 1 \quad \text{and} \quad \ell_k^{(n)}\left(x_i^{(n)}\right) = 0 \qquad \text{for} \quad 1 \le i \le n, \quad i \ne k.$$

Is it true that there is a point group for which for every continuous $f(x)$ there always is a point x_0 for which there is convergence of the Lagrange interpolation polynomials

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though (2) holds, i.e., divergence would be possible? With Vértesi we conjecture that the answer is yes.

Grünwald and Macinkiewicz showed 60 years ago that for the roots of the Chebyshev polynomials there is a continuous function for which everwhere divergence is possible. The conjecture of Vértesi and myself would show that there is a point group for which the Lagrange interpolation is not completely bad.

Problem III. Halász and I proved (Colloq. Math. Soc. János Bolyai, **58**, North Holland, 1991, 263–374) that there is a continuous function $f(x)$ for which the arithmetic means of the Lagrange interpolation polynomials taken at the Chebyshev abcissas diverge almost everywhere.

We have two problems: is there a continuous function for which the arithmetic means of the Lagrange interpolation polynomials taken at the roots of the Chebyshev polynomials can diverge everywhere; and can there be divergence almost everywhere of the arithmetic means taken at a general point group, which of course would strengthen the result of Erdős and Vértesi.

Also, what happens for more general summability methods?

Problem IV. Let $x_i^{(n)}$, $1 \le i \le n$, $1 \le n < \infty$, $-1 \le x_1^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} \le 1$ be an arbitrary point group. Is it true that there is an x_0, $-1 < x_0 < 1$, for which for infinitely many n

$$\sum_{k=1}^{n} \left| \ell_k^{(n)}(x_0) \right| > \frac{2}{\pi} \log n - c \,? \tag{3}$$

Perhaps, in fact, (3) holds for almost all x_0.

Problem V. Fejér proved that for the roots of the integral of the Legendre polynomials

$$\left| \sum_{k=1}^{n} \ell_k^2(x) \right| \le 1$$

everywhere in $(-1, +1)$. I conjectured that

$$\int_{-1}^{+1} \left(\sum_{k=1}^{n} \ell_k^2(x) \right) dx$$

is minimal for the roots of the integral of the Legendre polynomial. This was disproved by Szabados.

We think that for any set $-1 = x_1 < \ldots < x_n = 1$.

$$\int_{-1}^{+1} \left(\sum_{k=1}^{n} \ell_k^2(x) \right) dx > 2 - \frac{c}{n}, \tag{4}$$

and perhaps the minimum differs from the values of

$$\int_{-1}^{+1} \left(\sum_{k=1}^{n} \ell_k^2(x) \right) dx$$

for the Legendre polynomial only by $o(1/n)$, but in our paper with Szabados, Varma and Vértesi (Studia Sci. Math. Hungar., **29**(1994), 55-60) we only could prove for the

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Legendre nodes that

$$\int_{-1}^{+1} \left(\sum_{k=1}^{n} \ell_k^2(x) \right) dx > 2 - \frac{c (\log n)^2}{n}.$$

Problem VI. In our old paper with Herzog and Piranian (Journal d'Analyse Math., **6**(1958), 125–148) we stated many unsolved problems on polynomials. Below I only state a few of them.

Let $f_n(z)$ be a polynomial of degree n and highest coefficient 1. Is it true that the length of the lemniscate $|f_n(z)| = 1$ is longest for $f_n(z) = z^n - 1$? Very recently P. Borwein proved that the lemniscate has length $< cn$ for an absolute constant c. Our conjecture would imply that the length of the lemniscate is less than $2 + o(1)$.

Problem VII. Suppose $|z_i| \le 1$, $1 \le i \le n$. Consider the set

$$|f_n(z)| \le 1, \qquad f_n(z) = \prod_{i=1}^{n} |z - z_i|.$$

Denote this set by E_n. We observed that the area of E_n can tend to 0 as $n \to \infty$. Is it greater than $1/(\log n)^c$? In fact, estimate or determine the minimum of the area of E_n.

Problem VIII. Newman and I conjectured very long ago that if

$$f_n(y) = \sum_{i=0}^{n} \varepsilon_i y^i, \qquad \varepsilon_i = \pm 1,$$

is a polynomial of degree n, then there is an absolute constant c independent of n and the ε_i's for which

$$\max_{|z|=1} |f_n(z)| > (1 + c)\sqrt{n} \tag{5}$$

The Rudin-Shapiro polynomials show that (5), if true, is the best possible apart from the value of c.

There is a surprising result of Kahane (Bull. London Math. Soc., **12**(1980), 321–342): there are complex numbers $|\varepsilon_i| = 1$, $1 \le i \le n$ such that for every $|z| = 1$

$$n^{\frac{1}{2}} - n^{\frac{3}{10}} < |f_n(z)| < n^{\frac{1}{2}} + n^{\frac{3}{10}}.$$

Thus our conjecture with Newman fails if $\varepsilon_i = \pm 1$ is replaced by $|\varepsilon_i| = 1$. Perhaps if $n_1 < n_2 \ldots < n_k$ is any sequence of integers, then

$$\max_{|z|=1} \left| \sum_{i=1}^{k} \varepsilon_i z^{n_i} \right| > (1 + c) k^{\frac{1}{2}} \tag{6}$$

holds. (6) of course would be much stronger than (5). (100 dollars for a proof or disproof of (5).)

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