Research Problems

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Two Problems on Interpolation

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Interpolation by polynomials is probably the oldest profession in Approximation Theory.

It combines an obvious practical use of curve fitting with the elements of risk, insecurity, and intellectual stimulation. The latter is supplied by the famous Faber theorem.

Let $\Delta_n = 0 \le t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} \le 1$ be a partition of the interval $[0, 1]$. Let $\mathcal{F}P_n$ stand for a polynomial of degree at most n. And let $L(\Delta_n) : C([0, 1]) \to \mathcal{F}P_n$ be the usual Lagrange interpolation operator.

Theorem 1 (Faber). *For any sequence $\Delta_n \subset [0,1]$; $n = 0, 1, 2 \ldots$ there exists a continuous function $f \in C([0,1])$ such that $L(\Delta_n)f$ does not converge to f in the uniform norm.*

In contrast, if the real-valued function f is given, then one can easily find a sequence of partitions $\Delta_n \subset [0, 1]$ such that $L(\Delta_n)f$ converges to f. Indeed, let $b_n(f)$ be the polynomial of best approximation to f. Then $f - b_n(f)$ alternates $n + 2$ times and hence (by the Intermediate Value Theorem) has $n + 1$ zeroes. Pick those to be Δ_n. Then $L(\Delta_n)f = b_n(f)$ and $L(\Delta_n)f \to f$. Easy? Well, what happens if f is complex valued?

"Whoops, there it is!"

Problem I. Given a complex-valued function $f \in C([0, 1])$, does there exist a sequence $\Delta_n \subset [0, 1]$ such that

$$L(\Delta_n)f \to f \in C([0, 1])?$$

Equivalently, given *two* real-valued functions $f, g \in C([0, 1])$, does there exist *one* sequence $\Delta_n \subset [0, 1]$ such that

$$L(\Delta_n)f \to f \text{ and } L(\Delta_n)g \to g?$$

The answer is almost certainly "yes." It is proven to be "yes" if f/g is a rational function (cf. [2]) or if one of the functions is in Lip1 class and the other one is arbitrary ([3]).

The only other significant development known to us is the monetary reward of one thousand shekels promised by P. Erdős for the solution of this problem.

Another popular form of interpolation is the Newton interpolation. That is the case where we restrict our choices of Δ_n and insist that, as sets, $\Delta_n \subset \Delta_{n+1}$ for all n.

Now the same question for one real-valued function is:

Problem II. Given a real-valued function $f \in C([0, 1])$, does there exist a sequence $\Delta_n \subset [0, 1]$ with $\Delta_n \subset \Delta_{n+1}$ such that $L(\Delta_n)f \to f$?

We conjecture (with less certainty) that the answer is "No". That is, we believe that there exists a real-valued function $f \in C([0, 1])$ such that, for any infinite sequence of disjoint points in $[0, 1]$, the Newton interpolation series does not converge to f.

A good candidate for such a function is:

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ dist } \left(2^k x, \mathbf{N}\right) \in C([0, 1]).$$

A short list of partial results on this problem is as follows:

For a given $f \in C([0, 1])$, there exists a sequence $\Delta_n \subset [0, 1]$ with $\Delta_n \subset \Delta_{n+1}$ such that $L(\Delta_{n_k})f \to f$ for some subsequence n_k of integers (cf. [1]).

For $f(x) = |x|$, Peter Borwein informed us that there exist convergent Newton series on $C[-1, 1]$. Using his idea, Plamen Simeonov contracted such a series with $\Delta_{2n} = \{\pm 1 - q^k\}_{k=1}^{h}$, $q \in (0, 1)$.

Xin Li informed us that there exists a sequence $\Delta_n \subset \Delta_{n+1}$ such that $L(\Delta_n)f$ converge to f for every piecewise analytic f.

It is not hard to come up with a similar set of problems where the interval $[0, 1]$ is replaced by a more general set or the interpolation is replaced with another form of approximation or the function f is vector-valued or $\mathcal{F}P_n$ is replaced with a different subspace. Some of these problems are outlined in [1] and some we leave to the creative taste of the reader. Almost every reasonable variation on this theme seems to pose a challenge.

The results of [2] indicate that some topological properties of the domain and the range of the function f are essential and hence the topological tools are probably needed for their solutions. (We did use the Intermediate Value Theorem.)

We also believe that there is a deeper meaning behind our inability to deal with these questions.

The "meat" in the proof of the Faber Theorem is in the estimate of the norm $\|L(\Delta_n)\|$, which, as it turns out, tends to infinity at least as fast as $\log n$. The avalanche of papers in polynomial interpolation estimates $\|L(\Delta_n)\|$ and its various analogs. Thus we gain a good understanding for the quantity $(f(x) - (L(\Delta_n)f)(x))$ only for the "worst" (or "best") possible f.

It is peculiar that the estimate of $\|L(\Delta_n)f\|$ for an individual f is harder than the estimate of suprema of $\|L(\Delta_n)f\|$.

Here are some "philosophical" questions to which this leads:

Given a sequence of "random" Δ_n and a "random" f, what are the chances that $L(\Delta_n)f \to f$?

What is the relationship between $f(t_j^{(n)})$ and $\Delta_n = \{t_j^{(n)}\}$ which makes $L(\Delta_n)f$ resemble (or not) the function f?

For what *closed* subsets X of $C[0, 1]$ can one find a sequence of Δ_n (maybe with some restrictions) such that $L(\Delta_n)f \to f$ for all $f \in X$?

Over the last few years we circulated problems I and II in the mathematical community. They were met with interest. We received many comments and remarks about

The Bartlett Press, Inc. iss114˙r 15 · i · 1996 3:00 p.m. Page 515

these problems from our friends and colleagues. To all of them we are grateful. We are especially thankful to Peter Borwein and Ed Saff, who encouraged us to write this note. If anyone has any questions or comments related to this note, please contact us and we will be glad to supply you with what little information is known to us. GOOD LUCK.

References

1. B. Shekhtman, Some Simple Open Problems on Interpolation of Individual Functions. In: "Constructive Theory of Functions Varna 91," Sofia, 1992, pp. 259–268.
2. B. Shekhtman, On Interpolation of Individual Functions, J. of Computers and Mathematics, to appear.
3. B. Shekhtman, On Simultaneous Interpolation of Two Functions, in preparation.

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