Research Problems

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A Convergence Problem for Rational Interpolants with Preassigned Poles: Research Problem 96-1

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Let Ω be a domain in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ containing infinity, and f an analytic function in Ω given by its Taylor series at the point infinity. This series converges inside the maximal disk, centered at infinity, where f is analytic, and diverges outside the disk.

To overcome this incomplete form of convergence, Padé approximants were introduced. These are rational functions matching $f(z)$ at infinity with maximal possible degree; for the precise definition see, for instance, [2]. These approximants are very good for uniform approximation of some special classes of functions but, in general, we cannot guarantee convergence anywhere except at the point $z = \infty$ itself (see [3]). The cause for such unpredictable behavior of Padé approximants is mainly a chaotic, spurious distribution of their poles for large classes of functions.

The difficulties with spurious poles led to the definition of a type of rational approximants with preassigned poles, also known as Padé-type approximants.

Definition. Let $Q(z)$ be any polynomial and denote by $P(z)$ the polynomial part of the Laurent expansion of $Q(z)f(z)$ at infinity. Then the rational function $\pi(z) = P(z)/Q(z)$ is called the *Padé-type approximant* (PTA) of f with preassigned denominator Q (or PTA with preassigned poles at the zeros of Q).

The definition means that

$$Q(z)f(z) - P(z) = \mathcal{O}(z^{-1}) \qquad \text{as} \quad z \to \infty.$$

$\pi(z)$ does not depend on the normalization of $Q(z)$ so we may assume that $Q(z)$ is monic. The poles of the PTAs are preassigned in the sense that we may choose $Q(z)$ ourselves. This gives a great freedom in determining these approximants.

Consider the following problem: if Ω is given, is it possible to choose a sequence of monic polynomials Q_n, $n = 1, 2, \ldots$, where Q_n has degree n, so that, for any analytic function f in Ω, the PTAs of f with preassigned denominators Q_n converge to f in Ω, as n tends to infinity, uniformly on bounded closed subsets of Ω? If this is possible we say that $\{Q_n\}_0^\infty$ provides convergence in the PTA problem for Ω.

For domains Ω which are regular with respect to Dirichlet's problem we have the

following result, where $\mathrm{cap}(K)$ denotes the logarithmic capacity of K, and μ_n is the zero counting measure of Q_n which puts mass $1/n$ at each zero of Q_n counting multiplicity.

Theorem [1, Theorem 1]. *Let Ω be a regular domain in $\overline{\mathbf{C}}$ containing infinity, $K = \overline{\mathbf{C}} \setminus \Omega$, and $\{Q_n(z) = z^n + \cdots\}_{n=0}^{\infty}$ a sequence of polynomials such that the set of zeros of the polynomials in the sequence has no limit point in Ω. Then the following conditions are pairwise equivalent*:

(1) $\{Q_n\}_0^{\infty}$ *provides convergence in the PTA problem for* Ω.
(2) $\lim_{n\to\infty} \max_{z\in K} |Q_n(z)|^{1/n} = \mathrm{cap}(K)$.
(3) $\lim_{n\to\infty} |Q_n(z)|^{1/n} = \mathrm{cap}(K)\, e^{g_\Omega(z)}$, *uniformly on bounded closed subsets of* Ω, *where $g_\Omega(z)$ is Green's function of Ω with pole at infinity*.
(4) *The zeros of $\{Q_n\}$ have regular asymptotic distribution on K in the sense that every weak limit of $\{\mu_n\}_1^{\infty}$ is supported by K and its sweeping out onto $\partial\Omega = \partial K$ coincides with the equilibrium measure of K.*
(5) *For an arbitrary closed bounded set $F \subset \Omega$ there exists an open neighborhood O of K such that*

$$\lim_{n\to\infty} \sup \left\{ \left| \frac{Q_n(t)}{Q_n(z)} \right|,\ t \in O, z \in F \right\} = 0.$$

For domains Ω which are not regular but satisfies $\mathrm{cap}(K) > 0$, we have the following implications between the conditions (1)–(5) (see [1, Theorem 2]): (5) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4). In general (1) does not follow from (2), and (2) does not follow from (3) (see [1, Proposition 2]). The only open question in this chain of implications is whether (1) implies (5) for arbitrary domains Ω with $\mathrm{cap}(K) > 0$. A positive solution to this question is equivalent (see [1]) to the following more important open problem, which we formulate as a conjecture.

Conjecture. *For the existence of polynomials $\{Q_n\}_0^{\infty}$ providing convergence in the PTA problem for a domain Ω with* $\mathrm{cap}(K) > 0$, *it is necessary and sufficient that Ω is regular*.

The existence of polynomials providing convergence in the case of regular domains Ω follows from the theorem, since the Chebyshev polynomials of K satisfy condition (2). The open part of the conjecture is that there does not exist polynomials providing convergence for irregular domains. We have proved the conjecture for the case when $K = \overline{\mathbf{C}} \setminus \Omega$ has an isolated point and the polynomials Q_n have no zeros in Ω (see [1, Proposition 1]). We note that even in this case our proof is quite complicated.

References

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