Research Problems

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Painlevé Equations for Semiclassical Recurrence Coefficients: Research Problems 96-2

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Background. Orthonormal polynomials $p_n(x) = \gamma_n x^n + \cdots$ with recurrence relation

$$a_{n+1}p_{n+1}(x) = (x - b_n)p_n(x) - a_n p_{n-1}(x) \tag{1}$$

related to a semiclassical weight w,

$$\frac{w'(x)}{w(x)} = \frac{2V(x)}{W(x)}\,, \tag{2}$$

where V and W are polynomials, satisfy differential relations

$$Wp_n' = (\Omega_n - V)p_n - a_n\Theta_n p_{n-1}, \tag{3}$$

where Ω_n and Θ_n are polynomials of degrees $\leq m-1$ and $m-2$ (if degrees V and $W \leq m-1$ and m), with

$$\Omega_{n+1}(x) = (x - b_n)\Theta_n(x) - \Omega_n(x), \tag{4}$$

$$(x - b_n)(\Omega_{n+1}(x) - \Omega_n(x)) = W(x) + a_{n+1}^2\Theta_{n+1}(x) - a_n^2\Theta_n(x)$$

representing nonlinear recurrence relations for the a_n's and b_n's (Laguerre and followers, see [2], [7], and [12], to quote only three papers with publication dates in approximate arithmetic progression).

Statement. *If, in addition to* (2), *the weight w depends on a parameter t so that*

$$\frac{\partial w(x,t)/\partial t}{w(x,t)} = \frac{T(x,t)}{W(x,t)}, \tag{5}$$

where T and W are polynomials in x and t (*the same W, which will now be written $W(x,t)$*), *each recurrence coefficient $a_n^2(t)$ and $b_n(t)$ satisfies a nonlinear differential equation having Painlevé property* (*movable singular points are only poles*).

I first saw what is basically this statement in [3, pp. 399–402], and [5], where D. and G. Chudnovsky show how works of R. Fuchs on monodromy is relevant to some Padé approximation studies. G. Chudnovsky is quoted by Nuttall [11, bottom of p. 363] with respect to the simplest generalized Jacobi orthogonal polynomials. Monodromy

techniques related to special orthogonal polynomials are used, for instance in [1], [4], [5], and [8]. This Painlevé connection allows us to solve very important asymptotic problems [1], [3]–[6], [8].

I tried in [9] to show this statement for **generalized Jacobi weights**

$$w(x,t)=\begin{cases}0 & (x<x_1(t) \text{ or } x>x_m(t))\\ C_j\prod_{k=1}^{m}|x-x_k(t)|^{\alpha_k} & (x_j(t)<x<x_{j+1}(t),\ j=1,\dots,m-1),\end{cases} \tag{6}$$

where therefore $W(x,t)=\prod_1^m(x-x_k(t))$ and (2) and (5) become

$$\frac{w'(x,t)}{w(x,t)}=\sum_{k=1}^{m}\frac{\alpha_k}{x-x_k(t)} \quad \text{and} \quad \frac{\partial w(x,t)/\partial t}{w(x,t)}=-\sum_{k=1}^{m}\frac{\alpha_k\dot{x}_k(t)}{x-x_k(t)}\,. \tag{7}$$

(N.B. So the C_j's and the α_j's do *not* depend on t; note also that $\alpha_k=2V(x_k)/W'(x_k)$, $\Omega_n(x)=(n+(\sum_1^m\alpha_k)/2)x^{m-1}+\cdots$, $\Theta_n(x)=(2n+1+\sum_1^m\alpha_k)x^{m-2}+\cdots$.) A differential system involving a_n and b_n is then

$$\frac{\dot{a}_n}{a_n}=\frac{1}{2}\sum_{k=1}^{m}\frac{(\Theta_n(x_k)-\Theta_{n-1}(x_k))\dot{x}_k}{W'(x_k)}, \tag{8}$$

$$\dot{b}_n=\sum_{k=1}^{m}\frac{(\Omega_{n+1}(x_k)-\Omega_n(x_k))\dot{x}_k}{W'(x_k)}=\sum_{k=1}^{m}\frac{((x_k-b_n)\Theta_n(x_k)-2\Omega_n(x_k))\dot{x}_k}{W'(x_k)}, \tag{9}$$

$$\frac{\dot{\gamma}_n}{\gamma_n}=-\frac{1}{2}\sum_{k=1}^{m}\frac{\dot{x}_k\Theta_n(x_k)}{W'(x_k)}, \tag{10}$$

$$\frac{d}{dt}\frac{\Theta_n(x_j)}{W'(x_j)}=2\frac{\dot{\gamma}_n}{\gamma_n}\frac{\Theta_n(x_j)}{W'(x_j)}-2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_n(x_k)\Omega_n(x_j)-\Theta_n(x_j)\Omega_n(x_k)}{W'(x_j)W'(x_k)}, \tag{11}$$

$$\begin{aligned}\frac{d}{dt}\frac{\Theta_{n-1}(x_j)}{W'(x_j)}&\\ =-2\frac{\dot{\gamma}_{n-1}}{\gamma_{n-1}}\frac{\Theta_{n-1}(x_j)}{W'(x_j)}&+2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_{n-1}(x_k)\Omega_n(x_j)-\Theta_{n-1}(x_j)\Omega_n(x_k)}{W'(x_j)W'(x_k)},\end{aligned} \tag{12}$$

$$\frac{d}{dt}\frac{\Omega_n(x_j)}{W'(x_j)}=a_n^2\sum_{k\neq j}\frac{\dot{x}_j-\dot{x}_k}{x_j-x_k}\frac{\Theta_n(x_j)\Theta_{n-1}(x_k)-\Theta_n(x_k)\Theta_{n-1}(x_j)}{W'(x_j)W'(x_k)}. \tag{13}$$

See [10] for a self-containing proof of (8)–(13) using only orthogonal polynomials techniques.

For instance, if $m=3$, a Painlevé equation of the sixth kind can be found for $\Theta_n(x_1)$ [3], [5], [8], but I would like to see the equation for a_n and b_n:

Problem. Eliminate the Θ's and the Ω's in (8)–(13) to exhibit the scalar differential equations for $a_n(t)$ and $b_n(t)$, at least in the $m=3$ case (W of degree 3 in x and 1 in t), and in the even $m=5$ case ($w(x,t)=|x|^\alpha|x^2-x_1^2|^\beta|x^2-x_2^2|^\gamma$, $b_n=0$).

It should then be possible to treat other semiclassical weights as confluent cases, for instance, $\exp(-\alpha x)(x-x_1)^\beta(x-x_2)^\gamma = \lim_{N\to\infty}(1-\alpha x/N)^N(x-x_1)^\beta(x-x_2)^\gamma$, etc. Simpler Painlevé transcendents should then be recovered through the same confluence process applied to the Painlevé sixth transcendent. See, for instance, how the fourth Painlevé transcendent appears in [6] and [8].

Exercise. Show that the moment $\mu_n = \int_{\mathbf{R}} w(u)(u-x_1)^n\,du$ of (6) enters the solution of the following differential system as $\mu_n = \nu_{n,1}$, where

$$\nu_{n,j} = \int_{\mathbf{R}} w(u)\prod_1^m (u-x_k)^{\beta_k}(u-x_j)^{-1}\,du:$$

$$\dot\nu_{n,j} = \sum_{\substack{k=1\\k\neq j}}^m (\dot x_j - \dot x_k)(\alpha_k+\beta_k)\frac{\nu_{n,j}-\nu_{n,k}}{x_j-x_k}, \qquad j=1,\dots,m,$$

with $\beta_1 = n+1$, $\beta_2 = \cdots = \beta_m = 1$.

This shows that the moments of these generalized Jacobi weights are regular functions of t in any region where the $x_k(t)$'s are distinct. As $a_n^2(t)$ and $b_n(t)$ are ratios of determinants of moments, the movable singular points of their equations are expected to be only poles (the zeros of the denominator determinants).

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