Research Problems

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On the Convergence of the Bieberbach Polynomials Inside the Domain: Research Problems 97-1

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1. Let G be a domain bounded by a Jordan curve G, let $z_0 \in G$, and let f be the conformal map of G onto $\{w\colon |w| < r\}$ normalized by $f(z_0) = 0$, $f'(z_0) = 1$. The Bieberbach polynomials π_n are defined as the polynomials that minimize

$$\iint_G |P'(z)|^2\,dx\,dy$$

in the class of all polynomials P of the form $P(z) = z + a_2z^2 + \cdots + a_nz^n$. It is easy to see that they also minimize

$$\iint_G |f'(z) - P'(z)|^2\,dx\,dy$$

in that class of polynomials. If $\{p_k\}$ is the (normalized) system of orthogonal polynomials over G, the π_n can be expressed by

$$\pi'_{n+1}(z) = \sum_{k=0}^{n} \overline{p_k(0)}\,p_k(z) \Big/ \sum_{k=0}^{n} |p_k(0)|^2.$$

2. Concerning the convergence of the Bieberbach polynomials, we have always $\pi_n(z) \to f(z)$ $(n \to \infty)$, uniformly on compact subsets of G, and estimates for

$$E_nu := \max\{|f(z) - \pi_n(z)| : z \in G\}$$

have been given under various assumptions on ∂G; see [1], [2], and [4].

Question. How fast is the convergence of the π_n to f on compact subsets of G?

A result in [4, p. 87] indicates that convergence inside G can be faster than convergence on $\overline{G}$. But the question is more precisely: *Can geometric convergence take place on a closed disk $\overline{D} \subset G$?* Can it be that

$$(*) \qquad |f(z) - \pi_n(z)| = O(q^n) \quad \text{for } z \in K \text{ and some } q < 1?$$

Naturally, $(*)$ is true if f has a holomorphic extension to $\overline{G}$, in particular if ∂G is an analytic Jordan curve. We conjecture that $(*)$ holds *only* if f has a holomorphic extension to $\overline{G}$.

3. This conjecture is supported by two observations.

(a) If ∂G has an exterior angle $< \pi$ at a point $\zeta \in \partial G$, then according to a recent result of Shirokov and Totik [3], $(*)$ will hold for any polynomials π_n of degree $\leq n$ only if f is holomorphic at ζ, which is not possible for our mapping function.

(b) It is not difficult to see that the "unnormalized" polynomials

$$P_n(z) = \sum_{k=0}^{n} \overline{p_k(0)}\, p_k(z)$$

converge geometrically on a closed disk $K \subset G$ to the Bergman kernel $K(z, 0)$ if and only if the mapping function f has a holomorphic extension to $\overline{G}$.

References

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