Research Problems

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Conjectures Around the Baker–Gammel–Wills Conjecture: Research Problems 97-2

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Abstract. The Baker–Gammel-Wills Conjecture states that if a function f is meromorphic in a unit disk $\mathbf{D}$, then there should, at least, exist an infinite subsequence $N \subseteq \mathbf{N}$ such that the subsequence of diagonal Padé approximants to f developed at the origin with degrees contained in N converges to f locally uniformly in $\mathbf{D}/\{\text{poles of } f\}$. Despite the fact that this conjecture may well be false in the general form as stated here, it nevertheless has drawn much interest and has influenced research in the theory of Padé approximation in several respects. In the present paper, six new conjectures about the convergence of diagonal Padé approximants are formulated that lead in the same direction as the Baker–Gammel–Wills Conjecture. However, they are more specific and they are based on partial results and theoretical considerations that make it rather probable that these new conjectures hold true.

1. Introduction

Padé approximants are the natural analogues of Taylor polynomials in the field of rational functions.

Definition 1. Let the function f be analytic at the origin. Then the *Padé approximant* $[m/n]$ of degree $m, n \in \mathbf{N}$ to f is defined as the rational function $[m/n] := P_{mn}/Q_{mn}$ with polynomials $P_{mn} \in \mathcal{P}_m$, $Q_{mn} \in \mathcal{P}_n$, $Q_{mn} \not\equiv 0$, satisfying the relation

$$(Q_{mn} f - P_{mn})(z) = O(z^{m+n+1}) \qquad \text{as} \quad z \to 0, \tag{1.1}$$

where $O(\cdot)$ denotes Landau's symbol big "oh" (see [BGM]).

Since poles of the function f should not cause special difficulties for rational approximation, it is in many respects natural (and convenient) to extend Definition 1 in such a way that f may have a pole at the origin. This can easily be done: relation (1.1) has to be replaced by

$$(Q_{mn} f - P_{mn})(z) = O(z^{m+n+1-\mathrm{ord}(f)}) \qquad \text{as} \quad z \to 0, \tag{1.2}$$

where $\mathrm{ord}(f)$ denotes the order of a possible pole of f at the origin. If f is analytic there, then $\mathrm{ord}(f) := 0$. Further, it has to be assumed that in (1.2) $n \geq \mathrm{ord}(f)$.

The theory of Padé approximants is very rich and has many connections to special methods in analysis and in numerical analysis (see [BGM]). In the present paper we are only concerned with the *convergence problem*, and in particular, the convergence of

diagonal Padé approximants $[n/n]$, $n \in \mathbf{N}$. A central place in the convergence theory is taken by the Baker–Gammel–Wills Conjecture [BGW].

Conjecture 1 (Baker–Gammel–Wills Conjecture). *If the function f is meromorphic in the unit disk* $\mathbf{D}$, *then there exists an infinite subsequence $N \subseteq \mathbf{N}$ such that*

$$[n/n](z) \to f(z) \qquad as \quad n \to \infty, \quad n \in N, \tag{1.3}$$

locally uniformly for $z \in \mathbf{D}\backslash\{$poles of $f\}$.

A slightly stronger version of Conjecture 1 demands that the convergence (1.3) holds locally uniformly for all $z \in \mathbf{D}$ in the spherical metric.

A counterexample in [W] shows that in general the convergence (1.3) does not hold for the whole sequence $\{[n/n]\}_{n\in\mathbf{N}}$. An inspection of this counterexample does not contradict Conjecture 1. Although no counterexample to Conjecture 1 is so far known, the conjecture is seen rather skeptically by many mathematicians (for a more detailed discussion of Conjecture 1 see [S4]).

2. Convergence in Capacity

In the present paper we consider two types of convergence: locally uniform convergence and convergence in capacity. The notion of convergence in capacity is here important not only since in this area progress has been made in the last 15 years (see [N1], [Po], [S1], [S2], and [GR]), but also because this notion is essential in the definition of spurious poles at the end of the present section. In the sequel by $\mathrm{cap}(\cdot)$ we denote the (logarithmic) capacity (for a definition see [La], [T], or [ST], Appendix I]).

Definition 2. A sequence of functions $f_n, n = 1, 2, \ldots$, is said to *converge in capacity* to f in a domain $D \subseteq \bar{\mathbf{C}}$ if for every $\varepsilon > 0$ and every compact set $V \subseteq D\backslash\infty$ we have

$$\lim_{n\to\infty} \mathrm{cap}\{z \in V \mid |(f_n - f)(z)| > \varepsilon\} = 0. \tag{2.1}$$

The big advantage of convergence in capacity in rational approximation is that it allows the approximants to have poles clustering in the convergence domain D, even if these poles do not correspond to poles of the function f. A basic result is the Nutall–Pommerenke theorem.

Theorem 1 ([N1], [P]). *Let the function f be meromorphic* (*and single-valued*) *in the domain $\bar{\mathbf{C}}\backslash E$ with $E \subseteq \bar{\mathbf{C}}$ compact and* $\mathrm{cap}(E) = 0$. *Then for every $\varepsilon > 0$ and every compact set $V \subseteq CC$ we have*

$$\lim_{n\to\infty} \mathrm{cap}\{z \in V \mid |(f - [n/n])(z)| > \varepsilon^n\} = 0. \tag{2.2}$$

Relation (2.2) is stronger than (2.1) since we have ε^n instead of ε. It follows from examples in [L1] and [R] that the assumption $\mathrm{cap}(E) = 0$ is essential in the Nuttall–Pommerenke theorem. However, these counterexamples only show that in general the full sequence $\{[n/n]\}_{n\in\mathbf{N}}$ will not converge in capacity in any subdomain of $\bar{\mathbf{C}}$ if the assumptions of Theorem 1 are not satisfied. Hence, the next conjecture is not excluded, and more than that, the analysis in [L2] and [Pa] gives some backing to the conjecture.

Conjecture 2. *If the function f is meromorphic in the unit disk* $\mathbf{D}$, *then there exists an infinite subsequence* $N \subseteq \mathbf{N}$ *such that the limit* (1.3) *holds true in capacity in the disk* $\mathbf{D}$.

In [M] it has been shown that convergence in capacity implies pointwise convergence quasi-everywhere for appropriately chosen subsequences. (A property is said to hold *quasi-everywhere* on a set $S \subseteq \mathbf{C}$ if it holds for all $z \in S$ with possible exceptions on a subset of outer capacity zero). Using this result, Conjecture 2 implies that if the function f is meromorphic in the unit disk $\mathbf{D}$, then there exists an infinite subsequence $N \subseteq \mathbf{N}$ such that limit (1.3) holds true for quasi every $z \in \mathbf{D}$.

Further it seems probable that in analogy with the generalization of Parfenov's result in [Pa] to a multiple connected domain D by Prokhorov in [Pr] that a proof of Conjecture 2 may open the way to a proof of the following more general conjecture.

Conjecture 3. *Let D be a domain in $\bar{\mathbf{C}}$, $z_0 \in D$ an arbitrary point, f a function meromorphic in D, and $[n/n]$ the Padé approximant to f developed at z_0. Then there exists an infinite subsequence $N \subseteq \mathbf{N}$ such that the sequence $[n/n]$, $n \in N$, converge in capacity to f in D.*

We close this section by defining the spuriousness of poles of Padé approximants. It has already been mentioned earlier that in many situations locally uniform convergence of Padé approximants cannot be proved since the approximants have poles clustering in side of the potential convergence domain D, and that these poles do not correspond to poles of the function f to be approximated. The name "spurious" had been coined by G. Baker in the early 1960s. An earlier example of spurious poles of Padé approximants (in the second column of the Padé table) were constructed in [Pe, Chapter 10], and spurious poles of diagonal Padé approximants are contained implicitly in [D]. For a more detailed discussion see [S4].

A formal definition of spurious poles has to take into account that the spurious character of a pole of a Padé approximant $[n/n]$ can be determined only by an asymptotic consideration ($n \to \infty$).

Definition 3. Let $[n/n]$, $n = 1, 2, \ldots$, be a sequence of diagonal Padé approximants to a function f and $N \subseteq \mathbf{N}$ an infinite subsequence. *Spurious poles* are defined in two situations:

(i) Let the approximant $[n/n]$ have a pole at $z_n \in \bar{\mathbf{C}}$ for each $n \in N$, and assume that $z_n \to z_0$ as $n \to \infty$, $n \in N$. If the function f is analytic at z_0 and if the approximants $[n/n]$ converge to f in capacity in a neighborhood of z_0, then the pole of the approximant $[n/n]$ at z_n, $n \in N$, is called *spurious*. If $z_0 = \infty$, then the convergence $z_n \to z_0$ has to be understood in the spherical metric.

(ii) Let the function f have a pole of order k_0 at $z_0 \in \bar{\mathbf{C}}$, and let the total order of the poles of the approximant $[n/n]$ near z_0 be $k_1 = k_{1n} > k_0$ for each $n \in N$, i.e., $[n/n]$ has poles at points z_{nj}, $j = 1, \ldots, m_n$, with total order k_1 and $z_{nj_n} \to z_0$ as $n \to \infty$, $n \in N$, for any selection $j_n \in \{1, \ldots, m_n\}$, $n \in N$. If the approximants $[n/n]$, $n \in N$, converge in capacity to f in a neighborhood of z_0, then poles of order $k_1 - k_0$ of the poles of the approximant $[n/n]$, $n \in N$, near z_0 are *spurious*.

The assumed convergence in capacity implies that spurious poles appear paired with zeros of the approximant $[n/n]$, and the distance between spurious poles and the corresponding zeros tend to zero as $n \to \infty, n \in \mathbf{N}$. For a more detailed discussion, including a discussion of the effect of pole-clearing, see [S4, Theorem 3.6] and [S3, Theorem 3.7].

3. Padé Approximants to Algebraic Functions

After some preparations four conjectures will be formulated. Three of them are new, and one has been formulated by J. Nuttall about 15 years ago.

Throughout this last section we assume that f is an algebraic function, i.e., there exist $(m+1)$-polynomials P_j, $j = 0, \ldots, m$, such that

$$R(z, f) = P_0(z) + P_1(z) f + \cdots + P_m(z) f^m \equiv 0. \tag{3.1}$$

In [S2] and [S3] results about the convergence in capacity of diagonal Padé approximants have been proved that cover algebraic functions. We repeat two results in a specialized form.

Theorem 2 ([S2, Theorem 1.1]). *Let the function f be algebraic and having no branch point at the origin. Then there exists a domain (the convergence domain) $D = D_f \subseteq \bar{\mathbf{C}}$ such that the diagonal Padé approximants $[n/n], n = 1, 2, \ldots$, converge to f in capacity in D, and the sequence $\{[n/n], n = 1, 2, \ldots\}$, does not converge in capacity to f in any domain $\tilde{D} \subseteq \bar{\mathbf{C}}$ with $\operatorname{cap}(\tilde{D} \backslash D) > 0$.*

The convergence domain D of Theorem 2 can be characterized independently of the convergence problem (see [S2, Theorem 1.3]).

Theorem 3. *The convergence domain $D = D_f$ of Theorem 2 is uniquely determined up to a set of capacity zero by the following two conditions*:

(i) *D is a subdomain of $\bar{\mathbf{C}}$ with $0 \in D$, and f has a single-valued meromorphic continuation throughout D.*

(ii) $\operatorname{cap}([\bar{\mathbf{C}} \backslash D]^{-1}) = \inf_{\tilde{D}} \operatorname{cap}([\bar{\mathbf{C}} \backslash \tilde{D}]^{-1})$ *with $S^{-1} := \{z \in \bar{\mathbf{C}} \mid 1/z \in S\}$ and the infimum extends over all domains $\tilde{D} \subseteq \bar{\mathbf{C}}$ satisfying* (i).

Much more than disks, the domain D of Theorems 2 and 3 can be considered as the natural convergence domain for diagonal Padé approximants to an algebraic function f (perhaps also for other types of functions). Results obtained in [S3] and investigations in [N2] suggest that it should be possible to describe the occurrence and distribution of spurious poles sufficiently precisely in case of algebraic functions. The first conjecture in the present section has been formulated by J. Nuttall (private communication).

Conjecture 4 (J. Nuttall). *Let the function f be algebraic and having no branch point at the origin. Then there exists a finite upper bound for the number of spurious poles (in the sense of total order) that each diagonal Padé approximant $[n/n], n \in \mathbf{N}$, can have.*

In Conjecture 4 only the existence of an upper bound is demanded, of course it will also be interesting to find a sharp value for this bound. In most cases this value should

be the genus of the Riemann surface of the function f (for a more detailed discussion see [S4]).

If Conjecture 4 can be proved, then it is probable that the proof will also give some insight in the distribution of spurious poles, and it is conjectured that this insight will provide some orientation toward a proof of the next conjecture.

Conjecture 5. *Let the function f be algebraic and having no branch point at the origin. Further let $D = D_f \subseteq \bar{\mathbf{C}}$ be the convergence domain introduced in Theorem* 2, *and characterized independently in Theorem* 3. *Then there exists an infinite subsequence $N \subseteq \mathbf{N}$ such that*

$$[n/n](z) \to f(z) \qquad as \quad n \to \infty, \quad n \in N, \tag{3.2}$$

locally uniformly for $z \in D$ in the spherical metric.

Using properties of the capacity (see [S3], Corollary 3.9]) one can deduce from Conjecture 5 the next conjecture, which is therefore a special case of Conjecture 5, but at the same time also a special case of Conjecture 1.

Conjecture 6. *Let the function f be algebraic and meromorphic in the unit disk $\mathbf{D}$. Then there exists an infinite subsequence $N \subseteq \mathbf{N}$ such that the limit* (3.2) *holds true locally uniformly for $z \in \mathbf{D} \backslash \{$poles of $f\}$.*

Partial results leading in the direction of Conjecture 5 have been proved for hyperelliptical functions in [S3, Theorem 3.8]. However, in these results there are some restrictions, which may be overcome, and which then would lead to a proof of the next conjecture, which is the most special one in the present paper.

Conjecture 7. *Let the function f be hyperelliptic and meromorphic at the origin, and further let $D = D_f$ be the convergence domain of Theorems* 2 *and* 3. *Then there exists an infinite subsequence $N \subseteq \mathbf{N}$ such that the limit* (3.3) *holds true locally uniformly for $z \in D$ in the spherical metric.*

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