Newman's Proof of PNT

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The Famous Prime Number Theorem

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x} \qquad as \qquad x \to \infty$$

was studied already by Legendre and Gauss. However, it took a hundred years before the first proofs appeared, one by Hadamard and one by de la Vallee Poussin (1896).

Technique used in the past :

Weiner's Tauberian theory of Fourier integrals
(Ikehara-Weiner Theorem)

Ikehara-Weiner Theorem

Let $f(x)$ be non negative, non decreasing on $[1, \infty)$ and such that the Mellin transform

$$g_0(s) = \int_1^{\infty} x^{-s} df(x) = -f(1) + s \int_1^{\infty} f(x) x^{-s-1} dx$$

exists for $\Re(s) > 1$. Suppose that for some constant c , the function ($g_0(s) - \frac{c}{s-1}$) has a continuous extension to the closed half plane $\Re(s) \geq 1$. Then

$$\frac{f(x)}{x} \longrightarrow c \quad as \quad x \longrightarrow \infty$$

D. J. Newman

Outline of Newman's Proof

1. Auxiliary Tauberian Theorem (Complex Integration)
2. Corollary - A Poor Man's Version of Ikehara-Weiner Theorem
3. Corollary $\Rightarrow$ Prime Number Theorem

Auxiliary Tauberian theorem

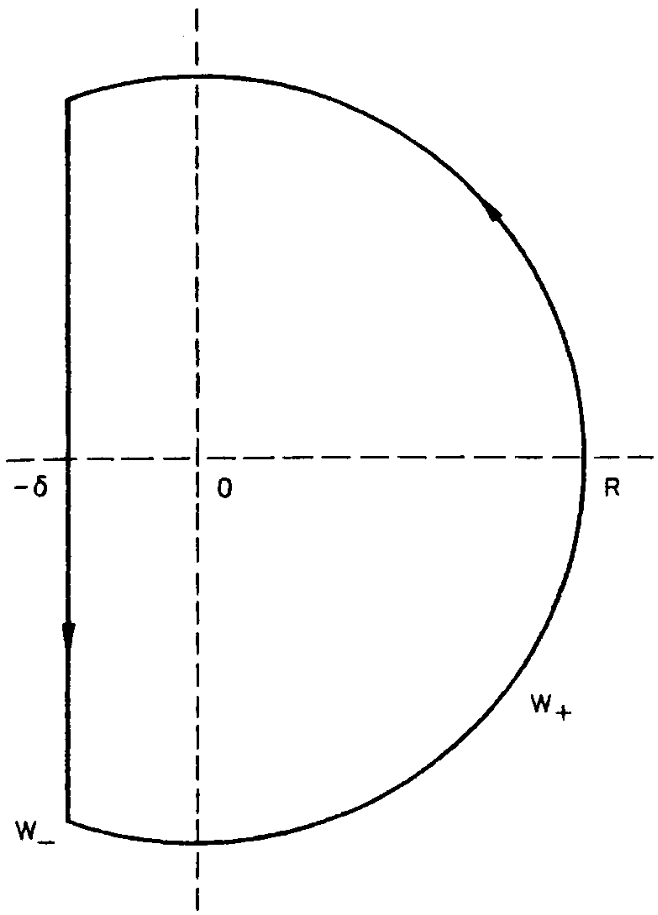
Let $F(t)$ be bounded on $(0, \infty)$ and integrable over every finite subinterval, so that the Laplace transform

$$G(z) = \int_0^\infty F(t) e^{-zt} dt$$

is well-defined and analytic throughout the open half-plane $\Re(z) > 0$. Suppose that G(z) can be continued analytically to a neighborhood of every point on the imaginary axis. Then

$$\int_0^\infty F(t) dt$$

exists as an improper intergral (and is equal to G(0))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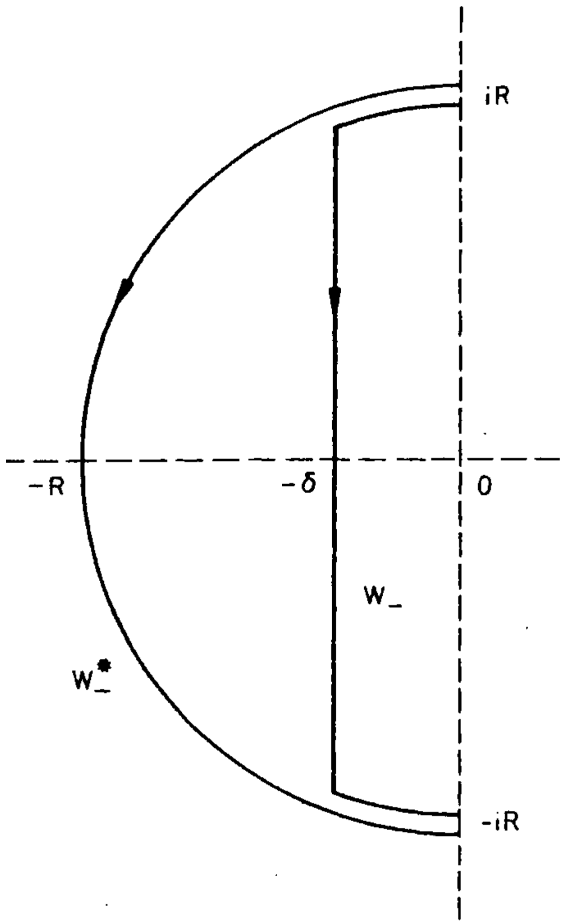
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We want to establish the following :-

Corollary to the auxiliary theorem

Let $f(x)$ be non negative, non decreasing and $O(x)$ on $[1, \infty)$, so that its Mellin transform

$$g(s) = s \int_1^\infty f(x) x^{-s-1} dx$$

is well-defined and analytic throughout the half-plane $\Re(s) > 1$. Suppose that for some constant c , the function ($g(s) - \frac{c}{s-1}$) can be continued analytically to a neighbourhood of every point on the line $\Re(s) = 1$. Then

$$\frac{f(x)}{x} \longrightarrow c \quad as \quad x \longrightarrow \infty$$

Corollary $\Rightarrow$ Prime Number theorem

One takes $f(x) = \psi(x)$ where $\psi(x)$ is that well-known function from prime number theory,

$$\psi(x) = \sum_{p^m \leq x} \log p$$

It is a simple fact that $\pi(x) = O(\frac{x}{\log x})$ or equivalently, $\psi(x) = O(x)$.
Thus $f(x)$ is as the corollary wants it.

What about its Melin transform $g(s)$?
A standard calculation based on the Euler product shows that

$$g(s) = -\frac{\zeta^{'}(s)}{\zeta(s)}, \quad \Re(s) > 1$$

Since $\zeta(s)$ behaves like $\frac{1}{s-1}$ around $s = 1$, the same is true for $g(s)$.Hence
$g(s) - \frac{1}{s-1}$ has an analytic continuation to a neighbourhood of the closed half plane $\Re(s) \geq 1$ The conclusion of the corollary now gives us

$$\frac{\psi(x)}{x} \rightarrow 1 \quad as \quad x \rightarrow \infty$$

We begin with the necessary facts about the zeta function.

B.1. Analytic continuation of $\zeta(s)$. Simple transformations show that for Re $s > 2$

$$\zeta(s) = \sum_1^\infty \frac{n}{n^s} - \sum_1^\infty \frac{n-1}{n^s} = \sum_1^\infty \frac{n}{n^s} - \sum_1^\infty \frac{n}{(n+1)^s} = \sum_1^\infty n\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} = \sum_1^\infty ns \int_n^{n+1} x^{-s-1}dx = s\sum_1^\infty \int_n^{n+1} [x]x^{-s-1}dx =$$

$$= s\int_1^\infty [x]x^{-s-1}dx, \tag{B.1}$$

where $[x]$ denotes the largest integer $\leqslant x$. Since first and final member are analytic for Re $s > 1$, the integral formula holds throughout that half-plane.

It is reasonable to compare the integral with

$$s\int_1^\infty x \cdot x^{-s-1}dx = \frac{s}{s-1} = 1 + \frac{1}{s-1}. \tag{B.2}$$

Combination of (B.1) and (B.2) gives

$$\zeta(s) - \frac{1}{s-1} = 1 + s\int_1^\infty ([x] - x)x^{-s-1}dx. \tag{B.3}$$

The new integral converges and represents an analytic function throughout the half-plane Re $s > 0$. Thus (B.3) provides an analytic continuation of the left-hand side to that half-plane.

B.2. Non-vanishing of $\zeta(s)$ *for* Re $s \geqslant 1$. The Euler product in (1.2) shows that $\zeta(s) \neq 0$ for Re $s > 1$. For Re $s = 1$ we will use Mertens's clever proof of 1898. The key fact is the inequality

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geqslant 0, \quad \theta \text{ real.} \tag{B.4}$$

Suppose that $\zeta(1 + ib)$ would be equal to 0, where b is real and $\neq 0$. Then the auxiliary analytic function

$$\varphi(s) = \zeta^3(s)\zeta^4(s + ib)\zeta(s + 2ib)$$

would have a zero for $s = 1$: the pole of $\zeta^3(s)$ could not cancel the zero of $\zeta^4(s + ib)$. It would follow that

$$\log|\varphi(s)| \to -\infty \quad \text{as} \quad s \to 1. \tag{B.5}$$

We now take s *real* and > 1. By the Euler product,

$$\log|\zeta(s+it)| = -\mathrm{Re}\sum_{p}\log(1-p^{-s-it}) = \mathrm{Re}\sum_{p}\{p^{-s-it}+\frac{1}{2}(p^2)^{-s-it}+\frac{1}{3}(p^3)^{-s-it}+\ldots\} = \mathrm{Re}\sum_{1}^{\infty}a_n n^{-s-it} \quad \text{with} \quad a_n \geqslant 0.$$

Thus

$$\log|\varphi(s)| = \mathrm{Re}\sum_{1}^{\infty}a_n n^{-s}(3+4n^{-ib}+n^{-2ib}) = \sum_{1}^{\infty}a_n n^{-s}\{3+4\cos(b\log n)+\cos(2b\log n)\} \geqslant 0$$

because of (B.4), contradicting (B.5).

B.3. Representations for $\zeta'(s)/\zeta(s)$. Logarithmic differentiation of the Euler product in (1.2) gives

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{p^{-s}}{1-p^{-s}} \log p = \sum_p (p^{-s} + p^{-2s} + \ldots) \log p = \sum_1^{\infty} \Lambda(n) n^{-s}, \tag{B.6}$$

where $\Lambda(n)$ is the von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \\ 0 & \text{if } n \text{ is not a prime power.} \end{cases}$$

The corresponding partial sum function is equal to $\psi(x)$:

$$\psi(x) = \sum_{p^m \leqslant x} \log p = \sum_{n \leqslant x} \Lambda(n).$$

Proceeding as in (B.1), the series (B.6) leads to the integral representation

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \psi(x) x^{-s-1} dx, \quad \operatorname{Re} s > 1.$$

The integral converges and is analytic for Re $s > 1$ since by (B.7), $\psi(x) \leqslant x \log x$.

B.4. Relation between $\psi(x)$ and $\pi(x)$. By (B.7), $\psi(x)$ counts $\log p$ (for fixed p) as many times as there are powers $p^m \leqslant x$, hence

$$\psi(x) = \sum_{p \leqslant x} \left[\frac{\log x}{\log p}\right] \log p \leqslant \log x \sum_{p \leqslant x} 1 = \pi(x) \log x. \tag{B.9}$$

On the other hand, when $1 < y < x$,

$$\pi(x) = \pi(y) + \sum_{y<p\leqslant x} 1 \leqslant \pi(y) + \sum_{y<p\leqslant x} \frac{\log p}{\log y} < y + \frac{\psi(x)}{\log y}.$$

Taking $y = x/\log^2 x$ one thus finds that

$$\pi(x)\frac{\log x}{x} < \frac{1}{\log x} + \frac{\psi(x)}{x}\,\frac{\log x}{\log x - 2\log\log x}. \tag{B.10}$$

Combination of (B.9) and (B.10) shows that

$$\lim \pi(x)\frac{\log x}{x} = 1 \quad \text{if and only if} \quad \lim \frac{\psi(x)}{x} = 1. \tag{B.11}$$

We finally indicate a standard proof of the estimate

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$$\psi(x) = 0(x). \tag{B.12}$$

For positive integral n, the binomial coefficient $\binom{2n}{n}$ must be divisible by all primes p on $(n, 2n]$. Hence

$$\prod_{n<p\leqslant 2n} p \leqslant \binom{2n}{n} < 2^{2n},$$

so that

$$\sum_{2^{k-1}<p\leqslant 2^k} \log p \leqslant 2^k \log 2.$$

It follows that

$$\sum_{p\leqslant 2^k} \log p \leqslant (2^k + 2^{k-1} + \ldots + 1) \log 2 < 2^{k+1} \log 2$$

and hence there is a constant C such that

$$\sum_{p\leqslant x} \log p \leqslant Cx.$$

Since the prime powers higher than the first contribute at most a term $0(x^{1/2+\epsilon})$ to $\psi(x)$, inequality (B.12) follows.