Gauss-Legendre Quadrature

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Abstract. It is known that gaussian quadrature is one the best numerical methods for computing definite integrals of analytic functions. For example, consider using 101 evaluation points to find π from $\int_0^1 4\left(1+x^2\right)^{-1} dx$. If we use a closed type Newton-Cotes formula, which just integrates the interpolating polynomial which agrees with the integrand at the 101 equally spaced nodes $[0.00, 0.01, 0.02, ..., 1.00]$ then we can compute the integral to 54 significant digits. If we use the 101 point gaussian quadrature formula, then we can compute the integral to 134 significant digits. The evaluation points for an n point Gauss-Legendre quadrature formula are the n roots of the n^{th} Legendre polynomial. One of the drawbacks to using gaussian quadrature, is the problem of computing these roots. We will compare several methods for computing the nodes and weights for an n point Gauss-Legendre quadrature formula at various precisions. As an application , consider the problem of numerically computing a truncated fourier series of a non-periodic analytic function. In this case gaussian quadrature can be faster than the fast fourier transform based algorithm, if we only want a few trigonometric coefficients to high precision. If we compute the fourier series for $\exp(-x)$ on the interval $[0,1]$ up to the terms $\cos(2\pi 2^{10} x)$ and $\sin(2\pi 2^{10} x)$ then the 2048 point gaussian quadrature formula gives the coefficients to 32 digits to the right of the decimal point. Now compare this with an FFT based algorithm. The error in a 2^m point FFT algorithm is $O(2^{-2m})$, so we would need about 2^{27} evaluation points for only double precision (16 digit) accuracy and about 2^{54} evaluation points for quadruple precision (32 digit) accuracy.

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