Optimizing and Parallelizing the Modular GCD Algorithm

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This is joint work with Matthew Gibson
Compute $G = \gcd(A, B)$ in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. 
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Compute $G$ modulo primes $p_1, p_2, \ldots$ and recover $G$ using Chinese remaindering.
Problem

Compute $G = \text{GCD}(A, B)$ in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

Compute $G$ modulo primes $p_1, p_2, \ldots$ and recover $G$ using Chinese remaindering.

Let $\bar{A} = A/G$ and $\bar{B} = B/G$ be the cofactors.

Let $A = \sum_{i=0}^{da} a_i(x_2, \ldots, x_n)x_1^i$.

Let $B = \sum_{i=0}^{db} b_i(x_2, \ldots, x_n)x_1^i$.

Let $G = \sum_{i=0}^{dg} g_i(x_2, \ldots, x_n)x_1^i$.

Let $t = \max_{i=0}^{dg}$ terms $g_i$.

Interpolate $g_i(x_2, \ldots, x_n)$ modulo $p$ from $2t + \delta$ univariate images in $\mathbb{Z}_p[x_1]$ using smooth prime $p$. 
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Compute $G \mod p_1, p_2, \ldots$ and recover $G$ using Chinese remaindering.

Let $\bar{A} = A/G$ and $\bar{B} = B/G$ be the cofactors.

Let $A = \sum_{i=0}^{da} a_i(x_2, \ldots, x_n)x_1^i$. $CA = \text{GCD}(a_i(x_2, \ldots, x_n))$.

Let $B = \sum_{i=0}^{db} b_i(x_2, \ldots, x_n)x_1^i$. $CB = \text{GCD}(b_i(x_2, \ldots, x_n))$.

Let $G = \sum_{i=0}^{dg} g_i(x_2, \ldots, x_n)x_1^i$. $CG = \text{GCD}(CA, CB)$.

Let $t = \max_{i=0}^{dg} \# \text{terms } g_i$. $\Gamma = \text{GCD}(a_{da}, b_{db})$.

**Observation:** Most of the time is recursive GCDs in $n - 1$ variables and evaluation and interpolation not GCD in $\mathbb{Z}_p[x_1]$. 
Compute $G = \text{GCD}(A, B)$ in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

Let $A = \sum_i a_{i,j}(x_3, \ldots, x_n)x_1^i x_2^j$. \hspace{1cm} CA = \text{GCD}(a_i(x_3, \ldots, x_n)).$

Let $B = \sum_i b_{i,j}(x_3, \ldots, x_n)x_1^i x_2^j$. \hspace{1cm} CB = \text{GCD}(b_i(x_3, \ldots, x_n)).$

Let $G = \sum_i g_{i,j}(x_3, \ldots, x_n)x_1^i x_2^j$. \hspace{1cm} CG = \text{GCD}(CA, CB).$

Let $s = \max_{i,j} \# \text{terms } g_{i,j}$. \hspace{1cm} \Gamma = \text{GCD}(LC(A), LC(B)).$

Interpolate $g_i(x_3, \ldots, x_n)$ modulo $p$ from $2s + \delta$ bivariate images in $\mathbb{Z}_p[x_1, x_2]$ using smooth prime $p$ – increased cost but
Bivariate Images

Compute \( G = \text{GCD}(A, B) \) in \( \mathbb{Z}[x_1, x_2, \ldots, x_n] \).

Let \( A = \sum_i a_{i,j}(x_3, \ldots, x_n)x_1^ix_2^j \). \( CA = \text{GCD}(a_i(x_3, \ldots, x_n)) \).

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- Usually \( s \ll t \) which reduces evaluation and interpolation cost.
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- Usually \( s \ll t \) which reduces evaluation and interpolation cost.
- Usually \( CA, CB, \Gamma \) are smaller so easier to compute.
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- Increases parallelism in interpolation.

1. Optimize serial bivariate Gcd computation.
2. For $n > 2$ parallelized (Cilk C) evaluation and interpolation.
3. Benchmark against Maple and Magma.
Bivariate Gcd computation.

Input $A, B \in \mathbb{Z}_p[y][x]$. Output $G = \text{GCD}(A, B)$, $\bar{A}$ and $\bar{B}$.

**Trial division method. (Maple, Magma)**
Interpolate $y$ in $G$ from univariate images in $\mathbb{Z}_p[x]$ **incrementally** until $G(x, y)$ does not change.
Test if $G | A$ and $G | B$. If yes output $G$, $\bar{A} = A / G$, $\bar{B} = B / G$. 
Bivariate Gcd computation.

Input $A, B \in \mathbb{Z}_p[y][x]$. Output $G = \text{GCD}(A, B)$, $\bar{A}$ and $\bar{B}$.

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**Cofactor recovery method.** (Brown 1971)
Interpolate $y$ in $G, \bar{A}, \bar{B}$ from univariate images
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$ in $\mathbb{Z}_p[x]$.
After $k$ images we have

$$A - G\bar{A} \equiv 0 \pmod{M} \quad \text{and} \quad B - G\bar{B} \equiv 0 \pmod{M}$$

where $M = (y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_k)$.
Stop when $k > \max(\deg_y A, \deg_y B, \deg_y G\bar{A}, \deg_y G\bar{B})$. 
Cofactor recovery method for $\mathbb{Z}_p[y][x]$

Interpolate $y$ in $G, \bar{A}, \bar{B}$ from univariate images
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$ in $\mathbb{Z}_p[x]$ in batches until one of $G, \bar{A}, \bar{B}$ stabilizes.

**Case $G$ stabilizes**: obtain remaining images using univariate $\div$:
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$
thus replacing the Euclidean algorithm with an evaluation.
Bivariate Gcd optimization.

Cofactor recovery method for $\mathbb{Z}_p[y][x]$ 

Interpolate $y$ in $G, \bar{A}, \bar{B}$ from univariate images 

$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$ in $\mathbb{Z}_p[x]$ 

in batches until one of $G, \bar{A}, \bar{B}$ stabilizes.

Case $G$ stabilizes: obtain remaining images using univariate $\div$ 

$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$ 

thus replacing the Euclidean algorithm with an evaluation.

Case $\bar{A}$ stabilizes: obtain remaining images using univariate $\div$ 

$\bar{a}_i = \bar{A}(\alpha_i, x), g_i = A(\alpha_i, x)/\bar{a}_i, \bar{b}_i = B(\alpha_i, x)/g_i$ 

thus replacing the Euclidean algorithm with an evaluation.
Figure: Image Division Optimizations

- Brown’s Algorithm
- Classical Division Method
- Maple 18
- Early $G$ and $\bar{B}$ stabilization
For dense $A, B$ in $\mathbb{Z}_p[x_3][x_1, x_2]$ we parallelize evaluation of $A$ and $B$ in blocks of size $j$ using a FFT of size $j$, run the bivariate GCDs in parallel, and parallelize interpolation of $G, \tilde{A}, \tilde{B}$ in batches of coefficients.
Parallel experiments in Cilk C

For dense $A, B$ in $\mathbb{Z}_p[x_3][x_1, x_2]$ we parallelize evaluation of $A$ and $B$ in blocks of size $j$ using a FFT of size $j$, run the bivariate GCDs in parallel, and parallelize interpolation of $G, \tilde{A}, \tilde{B}$ in batches of coefficients.

The algorithm is recursive and needs a lot of pieces of memory. We allocate large blocks of memory and use it as a stack. Memory for each bivariate Gcd is all preallocated.
Benchmarks \( A, B \in \mathbb{Z}_p[x_1, x_2, x_3] \), \( \deg A = \deg B = 200 \).

**jude** 2 x E5-2680 v2 CPUs, 10 cores, 2.8 GHz (3.6 GHz turbo).

### Table: Real times in seconds, \( p = 2^{62} - 57, 1373701 \) terms

<table>
<thead>
<tr>
<th>( \deg(G) )</th>
<th>( \deg(\widetilde{A}) )</th>
<th>( -\text{opt} )</th>
<th>( EA% )</th>
<th>1</th>
<th>8</th>
<th>16</th>
<th>20</th>
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<td>13.10</td>
<td>11.9</td>
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<td>160</td>
<td>12.39</td>
<td>28.8</td>
<td>5.79</td>
<td>0.92</td>
<td>0.55</td>
<td>0.49</td>
<td>0.27</td>
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<tr>
<td>70</td>
<td>130</td>
<td>11.29</td>
<td>36.9</td>
<td>6.47</td>
<td>0.99</td>
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<td>41.0</td>
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<td>27.5</td>
<td>5.29</td>
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<td>0.46</td>
<td>0.40</td>
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<tr>
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<td>4.50</td>
<td>1.8</td>
<td>3.44</td>
<td>0.58</td>
<td>0.37</td>
<td>0.33</td>
<td>0.25</td>
</tr>
</tbody>
</table>
Benchmarks $A, B \in \mathbb{Z}_p[x_1, x_2, x_3]$, $\deg A = \deg B = 200$.
gaby two E5-2660 CPUs, 8 cores at 2.2 GHz (3.0 GHz turbo).

Table: Real times in seconds, $p = 2^{62} - 57$, inputs have 1373701 terms

<table>
<thead>
<tr>
<th>Deg</th>
<th>$\bar{A}$</th>
<th>$A \times B$</th>
<th>GCD</th>
<th>$A \times B$</th>
<th>GCD</th>
<th>$\text{MGCD}$</th>
<th>$#\text{CPUs}$</th>
<th>POLY</th>
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<td>25.65</td>
<td>267.16</td>
<td>920.48</td>
<td>159.71</td>
<td>7.75</td>
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<td>130</td>
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<td>439.80</td>
<td>1624.6</td>
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<tr>
<td>100</td>
<td>100</td>
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<tr>
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<td>436.11</td>
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<tr>
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<td>90.30</td>
<td>2229.8</td>
<td>4.69</td>
<td>1.29</td>
<td>0.47</td>
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</table>
Current work

Let \( G = \sum_{i=0}^{dg} g_i(x_2, \ldots, x_n)x_1^i. \)
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- Most of the time is evaluation: $O((\#A + \#B)t)$. 
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- Have parallelized evaluation in batches of points.
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Thank you for attending my talk. Questions?