Optimizing and and Parallelizing the Modular GCD Algorithm

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PASCO 2015, Bath, England
July 10, 2015

This is joint work with Matthew Gibson

Problem

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Let $\bar{A} = A/G$ and $\bar{B} = B/G$ be the cofactors.
Let $A = \sum_{i=0}^{da} a_i(x_2, ..., x_n) x_1^i$.
Let $B = \sum_{i=0}^{db} b_i(x_2, ..., x_n) x_1^i$.
Let $G = \sum_{i=0}^{dg} g_i(x_2, ..., x_n) x_1^i$.
Let $t = \max_{i=0}^{dg} \#terms\ g_i$.

Interpolate $g_i(x_2, ..., x_n)$ modulo p from $2t + \delta$ univariate images in $\mathbb{Z}_p[x_1]$ using smooth prime p.

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Let $A = \sum_{i=0}^{da} a_i(x_2, ..., x_n)x_1^i$. $CA = GCD(a_i(x_2, ..., x_n))$.
Let $B = \sum_{i=0}^{db} b_i(x_2, ..., x_n)x_1^i$. $CB = GCD(b_i(x_2, ..., x_n))$.
Let $G = \sum_{i=0}^{dg} g_i(x_2, ..., x_n)x_1^i$. $CG = GCD(CA, CB)$.
Let $t = \max_{i=0}^{dg} \#terms\ g_i$. $\Gamma = GCD(a_{da}, b_{db})$.

Observation: Most of the time is recursive GCDs in $n-1$ variables and evaluation and interpolation not GCD in $\mathbb{Z}_p[x_1]$.

Bivariate Images

Compute $G = \mathrm{GCD}(A, B)$ in $\mathbb{Z}[x_1, x_2, ..., x_n]$.

Let $A = \sum_i a_{i,j}(x_3, ..., x_n) x_1^i x_2^j$. $CA = GCD(a_i(x_3, ..., x_n))$.
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Let $s = \max_{i,j} \# terms\ g_{i,j}$. $\Gamma = GCD(LC(A), LC(B))$.

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1. Optimize serial bivariate Gcd computation.
2. For $n > 2$ parallelized (Cilk C) evaluation and interpolation.
3. Benchmark against Maple and Magma.

Bivariate Gcd computation.

Input $A, B \in \mathbb{Z}_p[y][x]$. Output $G = GCD(A, B)$, $\bar{A}$ and $\bar{B}$.

Trial division method. (Maple, Magma)
Interpolate y in G from univariate images in $\mathbb{Z}_p[x]$ **incrementally** until $G(x, y)$ does not change.
Test if $G|A$ and $G|B$. If yes output $G, \bar{A} = A/G, \bar{B} = B/G$.

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Cofactor recovery method. (Brown 1971)

Interpolate y in $G, \bar{A}, \bar{B}$ from univariate images
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$ in $\mathbb{Z}_p[x]$.
After k images we have

$$A - G\bar{A} \equiv 0 \pmod{M} \quad \text{and} \quad B - G\bar{B} \equiv 0 \pmod{M}$$

where $M = (y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_k)$.
Stop when $k > \max(\deg_y A, \deg_y B, \deg_y G\bar{A}, \deg_y G\bar{B})$.

Bivariate Gcd optimization.

Cofactor recovery method for $\mathbb{Z}_p[y][x]$

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in batches until one of $G, \bar{A}, \bar{B}$ **stabilizes**.

Case G stabilizes: obtain remaining images using univariate $\div$
$g_i = G(\alpha_i, x), \bar{a}_i = A(\alpha_i, x)/g_i, \bar{b}_i = B(\alpha_i, x)/g_i$
thus replacing the Euclidean algorithm with an evaluation.

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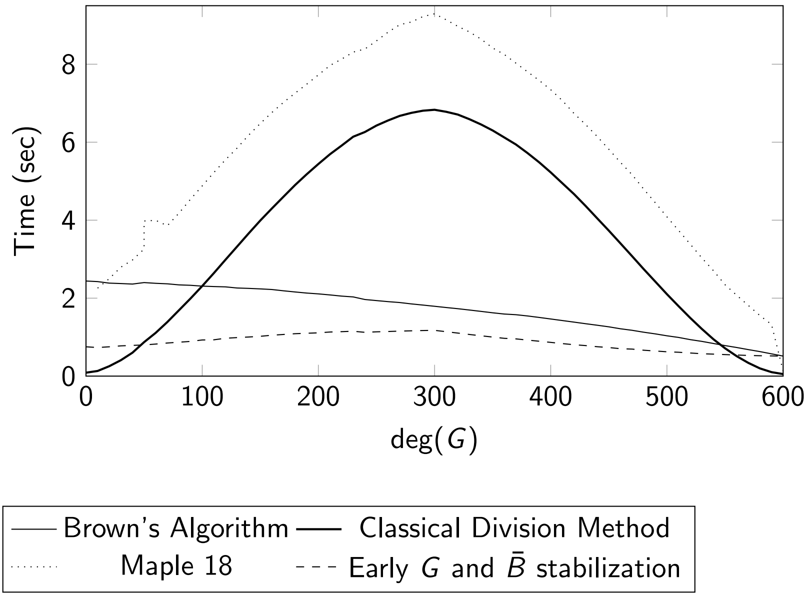
Case $\bar{A}$ stabilizes: obtain remaining images using univariate $\div$
$\bar{a}_i = \bar{A}(\alpha_i, x), g_i = A(\alpha_i, x)/\bar{a}_i, \bar{b}_i = B(\alpha_i, x)/g_i$
thus replacing the Euclidean algorithm with an evaluation.

Figure: Image Division Optimizations



Parallel experiments in Cilk C

For dense A, B in $\mathbb{Z}_p[x_3][x_1, x_2]$ we parallelize evaluation of A and B in blocks of size j using a FFT of size j, run the bivariate GCDs in parallel, and parallelize interpolation of $G, \bar{A}, \bar{B}$ in batches of coefficients.

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The algorithm is recursive and needs a lot of pieces of memory. We allocate large blocks of memory and use it as a stack. Memory for each bivariate Gcd is all preallocated.

Benchmarks $A, B \in \mathbb{Z}_p[x_1, x_2, x_3]$, $\deg A = \deg B = 200$.
jude 2 x E5-2680 v2 CPUs, 10 cores, 2.8 GHz (3.6 GHz turbo).

Table: Real times in seconds, $p = 2^{62} - 57$, 1373701 terms

$\deg(G)$	$\deg(\bar{A})$	−opt	$EA^{\%}$	1	8	16	20	Conv
10	190	13.10	11.9	4.79	0.84	0.54	0.48	0.37
40	160	12.39	28.8	5.79	0.92	0.55	0.49	0.27
70	130	11.29	36.9	6.47	0.99	0.56	0.49	0.21
100	100	9.93	41.0	6.72	1.00	0.57	0.50	0.18
130	70	8.38	27.5	5.29	0.80	0.46	0.40	0.18
160	40	6.52	14.4	4.16	0.66	0.39	0.34	0.20
190	10	4.50	1.8	3.44	0.58	0.37	0.33	0.25

Benchmarks $A, B \in \mathbb{Z}_p[x_1, x_2, x_3]$, $\deg A = \deg B = 200$.
gaby two E5-2660 CPUs, 8 cores at 2.2 GHz (3.0 GHz turbo).

Table: Real times in seconds, $p = 2^{62} - 57$, inputs have 1373701 terms

Deg		Maple		MagmaR		MGCD, #CPUs				POLY
G	$\bar{A}$	$A \times B$	GCD	$A \times B$	GCD	1	4	8	16	Conv
10	190	2.22	70.98	77.22	33.34	6.35	1.83	1.06	0.71	0.47
40	160	25.65	267.16	920.48	159.71	7.75	2.13	1.18	0.75	0.35
70	130	25.62	439.80	1624.6	462.09	8.72	2.35	1.27	0.75	0.28
100	100	25.43	453.27	1526.2	900.65	9.11	2.43	1.32	0.79	0.24
130	70	25.69	436.11	1559.2	14254.	7.11	1.92	1.04	0.62	0.23
160	40	25.44	282.04	934.45	7084.3	5.63	1.52	0.83	0.51	0.26
190	10	2.23	77.28	90.30	2229.8	4.69	1.29	0.74	0.47	0.32

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Thank you for attending my talk. Questions?