Tackling the difficult Kamke Ordinary Differential Equations (The method of the Great Bear)

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***4th April 2001**: The government of British Columbia endorsed the largest rainforest conservation measure in the history of North America: 20 critical pristine valleys in Canada's Great Bear Rainforest - 7 Million Acres of Ancient Rainforest - are to Be Protected.*

While preparing the statistics for Maple's **dsolve** performance in solving the approx.1,500 ODE examples from Kamke it turned apparent that Maple is still failing in solving some difficult ODEs. The topic of this talk is to present ***"a systematic approach for solving the ODE families underlying these more difficult ODE problems".*** The method involves a combination of techniques, including differential elimination and the solving of underlying PDE systems. In honor to the conservation of the Great Bear Rainforest I called this *"the approach of the Great Bear" :)*

The present performance with respect to the 576 first order ODE examples from Kamke is: *Solved* = $\frac{535}{547}$. What is so special in the remaining 12 unsolved equations? Two of these examples (number 25 and 144) are *Riccati type ODEs* and they are not solved due to failures in the linear ODE code. But the fix for this represents no mistery - the method missing is known. The problem is in the remaining 10 ODEs which are of no especific type.

Example: Kamke's ODE number 80; Classification: NONE

```
> _on;
```

y(x) will now be displayed as y

derivatives with respect to: x of functions of one variable will now be displayed with '

```
> declare(f(x));
```

f(x) will now be displayed as f

```
> ode[80] :=
diff(y(x),x)+f(x)*sin(y(x))+(1-diff(f(x),x))*cos(y(x))-diff(f(x),x)-1 = 0;
```

$$ode_{80} := y' + f\sin(y) + (1 - f')\cos(y) - f' - 1 = 0$$

Recall that: *Any first order ODE can be transformed into any other one by means of a change of variables of the form*

$$x = \mathrm{X}(t, \mathrm{u}(t)), \quad \mathrm{y}(x) = \mathrm{Y}(t, \mathrm{u}(t))$$

In fact all ODE solving methods amount to *"map a given (so-called: input) ODE into an ODE we know how to solve"*. For example: we know how to solve separable ODEs, so we map linear ODEs into separable. Then we map Bernoulli ODEs into linear, and so on.

For this example 80, Kamke shows a "solving" change of variables without further explanations on how to obtain it:

$$x = t, \quad \mathrm{y}(x) = 2 \arctan(\mathrm{u}(t))$$

The idea for a solving approach for these more difficult examples starts with the following consideration:

> *We use to search for transformations mapping difficult problems into linear first order ODEs, which we know how to solve. What about searching for transformations mapping into second order linear ODEs?*

In fact, all second order linear ODEs are equivalent to first order non-linear ODEs of *Riccati* type, i.e. of the form:

$$y' = c_2(x)\, y^2 + c_1(x)\, y + c_0(x)$$

where the right-hand-side is quadratic instead of linear. Although there exists no general method to solve all Riccati type ODEs, it happens that there are many methods for solving the corresponding second order linear ODEs Furthermore, a generalization of this idea would be

> *What about searching for transformations mapping a difficult problem into an Abel type ODE one?*

Indeed, Abel ODEs are of the form

$$y' = c_3(x)\, y^3 + c_2(x)\, y^2 + c_1(x)\, y + c_0(x)$$

so the right-hand-side is cubic insted of quadratic. The solving of Abel ODEs is more difficult than the solving of Riccati ODEs, but Maple has strong subroutines for tackling Abel ODEs too. With these ideas in mind the second step was

> *What would be an appropriate restriction in the general transformation* $x = \mathrm{F}(t, \mathrm{u}(t)),\ \mathrm{y}(x) = \mathrm{P}(t, \mathrm{u}(t))$, *such that the solving approach is "feasible"?*

The idea here was to start with something as simple as possible, and test whether or not the approach could solve some of these Kamke examples. The first idea, inspired in the transformation shown in Kamke, was

$$x = t, \qquad \mathrm{y}(x) = \mathrm{P}(\mathrm{u}(t))$$

Next step was to formulate the problem and run an experiment. So take ode_{80}:

```
> ode[80];
```

$$y' + f \sin(y) + (1 - f') \cos(y) - f' - 1 = 0$$

and change variables as mentioned:

```
> TR := {x=(t), y(x) = P(u(t))};
```

$$TR := \{x = t, y = \mathrm{P}(\mathrm{u}(t))\}$$

```
> o1 := ch(TR,ode[80]);
```

$$o1 := y' = \frac{-f\sin(\mathrm{P}(y)) - \cos(\mathrm{P}(y)) + \cos(\mathrm{P}(y))\, f' + f' + 1}{\mathrm{D}(P)(y)}$$

where in above I'm substituting $t = x$, $\mathrm{u}(t) = \mathrm{y}(x)$ in order to always work with x, $\mathrm{y}(x)$ as the variables. We want to determine $\mathrm{P}(y)$ such that ***o1*** is of *Riccati type; i.e. such that the right hand side of* ***o1*** *is quadratic in y.* We equate to zero the third derivative of the right-hand-side and simplify the expression, arriving at:

```
> e1 := simplify( diff( Yy(rhs(o1)), y$3), size) = 0;
```

$$e1 := \Bigg(((-1 - \cos(\mathrm{P}(y)))f' - 1 + \cos(\mathrm{P}(y)) + f\sin(\mathrm{P}(y)))\, P_y^{\;2}\, P_{y,y,y,y} - 6\, P_y \left(((-1 - \cos(\mathrm{P}(y)))f' - 1 + \cos(\mathrm{P}(y)) + f\sin(\mathrm{P}(y)))\, P_{y,y} - \frac{1}{3}(\sin(\mathrm{P}(y))f' - \sin(\mathrm{P}(y)) + f\cos(\mathrm{P}(y)))\, P_y^{\;2} \right) P_{y,y,y} + ((-6\cos(\mathrm{P}(y)) - 6)f' + 6f\sin(\mathrm{P}(y)) + 6\cos(\mathrm{P}(y)) - 6)\, P_{y,y}^{\;3} - 3\,(\sin(\mathrm{P}(y))f' - \sin(\mathrm{P}(y)) + f\cos(\mathrm{P}(y)))\, P_y^{\;2}\, P_{y,y}^{\;2} + P_y^{\;6}\,(\sin(\mathrm{P}(y))f' - \sin(\mathrm{P}(y)) + f\cos(\mathrm{P}(y))) \Bigg) \Big/ P_y^{\;4} = 0$$

Next step is not to solve the above for $\mathrm{P}(y)$... and for *arbitrary values of* $\mathrm{f}(x)$. At first sight, one would think that the problem above for $\mathrm{P}(y)$ is much more difficult than the ODE 80 we want to solve. However, the fact is that we are failing with ODE 80, and, nowadays, Maple has subroutines for tackling an problem like the one above for $\mathrm{P}(y)$ - this is basically the work developed during 1999 and 2000 here at CECM. A first "trial", concerning the existence of solution and triangularization for this problem for P would be:

```
> casesplit( e1, P, arbitrary=f, ivars={x,y}, RIF);
```

$$\left[P_{y,y,y} = -\frac{1}{2}\,\frac{-3\,P_{y,y}^{\;2} + P_y^{\;4}}{P_y},\ \sin(\mathrm{P}(y))^2 = -\cos(\mathrm{P}(y))^2 + 1 \right] \&\mathrm{where}\ [P_y \neq 0]$$

already showing that the approach is feasible: $\mathrm{P}(y)$ is given by the solution of a third order non-linear ODE, which however is of polynomial type and missing both y and $\mathrm{P}(y)$, so that two orders can be reduced immediately. In honor of the Canadian Great Bears, the solution for $\mathrm{P}(y)$ is then

```
> op([1,1],%);
```

$$P_{y,y,y} = -\frac{1}{2}\,\frac{-3\,P_{y,y}^{\;2} + P_y^{\;4}}{P_y}$$

```
> ans_P := ods(%);
```

$$ans\_P := \mathrm{P}(y) = 2\arctan\left(\frac{1}{4}\,\frac{2\,\_C1^2\, y + 2\,\_C2\,\_C1^2}{\_C1} \right) + \_C3$$

A simpler version is:

```
> a2 := eval(%,[_C3=0,_C2=0,_C1=2]);
```

$$a2 := \mathrm{P}(y) = 2\arctan(y)$$

That this solution is correct can be verfied by checking that it *indeed cancells the "third derivative of the rhs of the transformed ODE"*:

```
> simplify(  eval(e1, a2)  );
```

$$0 = 0$$

and in fact this is just the transformation shown in Kamke's book without further explanations. So let's use this value of P in the change of variables

```
> P=unapply(rhs(a2),y);
```

```
> TR1 := eval(TR,%);
```

$$TR1 := \{x = t,\ y = 2\arctan(\mathrm{u}(t))\}$$

Changing variables in *ode[80]*

```
> ch(TR1,ode[80]);
```

$$y' = \left(-\frac{1}{2} f \sin(2\arctan(y)) - \frac{1}{2}\cos(2\arctan(y)) + \frac{1}{2}\cos(2\arctan(y)) f' + \frac{1}{2} f' + \frac{1}{2}\right) y^2$$
$$-\frac{1}{2} f \sin(2\arctan(y)) - \frac{1}{2}\cos(2\arctan(y)) + \frac{1}{2}\cos(2\arctan(y)) f' + \frac{1}{2} f' + \frac{1}{2}$$

The resulting equation above is indeed of Riccati type. This can be verified by simplifyfing it via

```
> combine(convert(combine(convert(%,exp),exp),trig),trig);
```

$$y' = y^2 - y f + f'$$

and this Riccati ODE is solvable

```
> _in;a3 := ods(%,y(x));
Methods for first order ODEs:
Trying to isolate the derivative dy/dx...
Successful isolation of dy/dx
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
Chini's absolute invariant is:   1/4/diff(f(x),x)^3*(diff(diff(f(x),x),x)+2*f(x)*d
iff(f(x),x))^2
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE   diff(diff(Y(X),X),X) = -f(X)*diff(Y(X),X)-dif
f(f(X),X)*Y(X), Y(X)
      *** Sublevel 2 ***
      Methods for second order ODEs:
      Trying to isolate the derivative d^2Y/dX^2...
      Successful isolation of d^2Y/dX^2
      --- Trying classification methods ---
      -> Tackling the linear ODE "as given":
         trying a symmetry of the form [xi=0, eta=F(x)]
            <- BRANCH 3 successful. Symmetry is:   [0, 1/exp(Int(f(X),X))]
         <- linear_1 successful
   <- Riccati to 2nd Order successful
```

$$a3 := y = -\frac{e^{\left(\int f\,dx\right)}}{\_C1 + \int e^{\left(\int f\,dx\right)}\,dx} + f$$

So, by inverting the transformation used:

```
> gitr(TR1,{t,u(t)});
```

$$\{x = t, y = \tan\left(\frac{1}{2}\,\mathrm{u}(t)\right)\}$$

and changing variables in the solution above to the "Riccati simpler ODE"

```
> ch(%,a3);
```

$$\tan\left(\frac{1}{2}\,y\right) = -\frac{e^{\left(\int f\,dx\right)}}{\_C1 + \int e^{\left(\int f\,dx\right)}\,dx} + f$$

we get the solution to the original Kamke 80

```
> y(x)=solve(%,y(x));
```

$$y = 2\arctan\left(\frac{-e^{\left(\int f\,dx\right)} + \int e^{\left(\int f\,dx\right)}\,dx\,f + \_C1\,f}{\_C1 + \int e^{\left(\int f\,dx\right)}\,dx}\right)$$

This result can be tested

```
> odetest(%,ode[80]);
```

$$0$$

Example: Kamke's 81 is mappable into an Abel ODE

For that purpose, we just need to consider the transformation of the same form such that the right-hand-side is cubic, and so, the forth derivative of P(y) with respect to y whould be zero:

```
> ode[81] := diff(y(x),x)+2*tan(y(x))*tan(x)-1 = 0;
```

$$ode_{81} := y' + 2\tan(y)\tan(x) - 1 = 0$$

```
> TR := {x = t, y(x) = P(u(t))};
```

$$TR := \{y = \mathrm{P}(\mathrm{u}(t)), x = t\}$$

```
> o2 := ch(TR,ode[81]);
```

$$o2 := y' = \frac{-2\tan(\mathrm{P}(y))\tan(x) + 1}{\mathrm{D}(P)(y)}$$

Before searching for the transformation mapping into Abel, we show that there is ***no transformation of the form TR able to map this ODE 81 into a Riccati type ODE:*** for that we equate to zero the third derivative of the right-hand-side and simplify the expression:

```
> e2 := simplify( diff( Yy(rhs(o2)), y$3), size) = 0;
```

$$e2 := \Big(P_y^{\,2}\,(2\tan(\mathrm{P}(y))\tan(x)-1)\,P_{y,y,y,y}$$
$$+4\,P_y\left(\left(\left(-3\tan(\mathrm{P}(y))\tan(x)+\frac{3}{2}\right)P_{y,y}+\tan(x)\,P_y^{\,2}\,(1+\tan(\mathrm{P}(y))^2)\right)P_{y,y,y}\right.$$
$$+(12\tan(\mathrm{P}(y))\tan(x)-6)\,P_{y,y}^{\,3}-6\tan(x)\,P_y^{\,2}\,(1+\tan(\mathrm{P}(y))^2)\,P_{y,y}^{\,2}$$
$$-12\left(\tan(\mathrm{P}(y))^2+\frac{1}{3}\right)P_y^{\,6}\tan(x)\,(1+\tan(\mathrm{P}(y))^2)\Big)\Big/\;P_y^{\,4}=0$$

Now run a differential elimination process, showing that there is no solution to this system

```
> casesplit(e2, P, arbitrary=f, ivars={x,y}, RIF);
Warning: System is inconsistent
```

So try ***Abel*** now, by equating to zero the *fourth* derivative instead:

```
> e2 := simplify( diff( Yy(rhs(o2)), y$4), size) = 0;
```

$$e2 := \Big(P_y^{\,3}\,(2\tan(\mathrm{P}(y))\tan(x)-1)\,P_{y,y,y,y,y}$$
$$+((-16\tan(\mathrm{P}(y))\tan(x)+8)\,P_y^{\,2}\,P_{y,y}+6\tan(x)\,P_y^{\,4}\,(1+\tan(\mathrm{P}(y))^2))\,P_{y,y,y,y}$$
$$+(6-12\tan(\mathrm{P}(y))\tan(x))\,P_y^{\,2}\,P_{y,y,y}^{\,2}+8\,P_y\Big($$
$$\left(9\tan(\mathrm{P}(y))\tan(x)-\frac{9}{2}\right)P_{y,y}^{\,2}-\frac{7}{2}P_y^{\,2}\tan(x)\,(1+\tan(\mathrm{P}(y))^2)\,P_{y,y}+P_y^{\,4}\tan(\mathrm{P}(y))\tan(x)\,(1+\tan(\mathrm{P}(y))^2)$$
$$\Big)P_{y,y,y}+(-48\tan(\mathrm{P}(y))\tan(x)+24)\,P_{y,y}^{\,4}+24\,P_y^{\,2}\tan(x)\,(1+\tan(\mathrm{P}(y))^2)\,P_{y,y}^{\,3}$$
$$-12\,P_y^{\,4}\tan(\mathrm{P}(y))\tan(x)\,(1+\tan(\mathrm{P}(y))^2)\,P_{y,y}^{\,2}-24\,P_y^{\,6}\tan(x)\left(\tan(\mathrm{P}(y))^2+\frac{1}{3}\right)(1+\tan(\mathrm{P}(y))^2)\,P_{y,y}$$
$$-48\,P_y^{\,8}\tan(\mathrm{P}(y))\tan(x)\left(\tan(\mathrm{P}(y))^2+\frac{2}{3}\right)(1+\tan(\mathrm{P}(y))^2)\Big)\Big/\;P_y^{\,5}=0$$

The same differential elimination process now leads to a solution:

```
> casesplit(e2, P, arbitrary=f, ivars={x,y}, RIF);
```

$$[P_{y,y}=-2\,P_y^{\,2}\tan(\mathrm{P}(y))]\ \&\text{where}\ [P_y\neq 0,\ 1+\tan(x)^2\neq 0,\ 1+\tan(\mathrm{P}(y))^2\neq 0]$$

```
> op([1,1],%);
```

$$P_{y,y}=-2\,P_y^{\,2}\tan(\mathrm{P}(y))$$

So, in honor of the Great Bears, here is the transformation $\mathrm{P}(y)$ we are looking for

```
> dsolve(%);
```

$$\mathrm{P}(y)=\arctan(\_C1\,y+\_C2)$$

Again, a simpler form is obtained by taking $\_C2=0,\ \_C1=1$.

```
> eval(%,[_C1=1,_C2=0]);
```

$$\mathrm{P}(y)=\arctan(y)$$

```
> P=unapply(rhs(%),y);
```

$$P=\arctan$$

so that the transformation to be used is

```
> TR1 := eval(TR,%);
```

$$TR1 := \{y = \arctan(\mathrm{u}(t)),\ x = t\}$$

Changing variables, we arrive at the Abel ODE

```
> o4 := ch(TR1,ode[81]);
```

$$o4 := y' = -2\tan(x)\,y^3 + y^2 - 2\,y\tan(x) + 1$$

And, indeed, this Abel ODE is solvable by Maple in implicit form as

```
> _in;a4 := dsolve(%);
Methods for first order ODEs:
Trying to isolate the derivative dy/dx...
Successful isolation of dy/dx
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
The relative invariant s3 is:   10/3*tan(x)^2+20/27
The first absolute invariant s5^3/s3^5 is:   -2916*(-150*tan(x)^2-140+900*tan(x)^4
)^3/(90*tan(x)^2+20)^5
The second absolute invariant s3*s7/s5^2 is:   5/3*(9*tan(x)^2+2)*(-15*tan(x)^2-32
4*tan(x)^4+972*tan(x)^6+98)/(-15*tan(x)^2-14+90*tan(x)^4)^2
...checking Abel class AIL (45)
...checking Abel class AIL (310)
...checking Abel class AIR (36)
...checking Abel class AIL (301)
...checking Abel class AIL (1000)
...checking Abel class AIL (42)
...checking Abel class AIL (185)
inverse of the transformation solving the problem is:   {t = tan(x), u(t) = tan(x)
^4/(1+tan(x)^2)*y(x)-tan(x)^3/(1+tan(x)^2)}
<- Abel successful
```

$$a4 := \_C1 + \frac{\tan(x)}{\left(\left(\dfrac{1}{\tan(x)} + \dfrac{\tan(x)^2}{\dfrac{\tan(x)^4\,y}{1+\tan(x)^2} - \dfrac{\tan(x)^3}{1+\tan(x)^2}}\right)^2 + 1\right)^{(1/4)}} - \frac{1}{2}(y + \tan(x))$$
$$\left(-2 + \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{3}{2}\right], -\frac{(y+\tan(x))^2}{(y\tan(x)-1)^2}\right)\left(\frac{y^2 + 1 + \tan(x)^2\,y^2 + \tan(x)^2}{(y\tan(x)-1)^2}\right)^{(1/4)}\right) \Bigg/ \left(\right.$$
$$\left.(y\tan(x)-1)\left(\frac{y^2 + 1 + \tan(x)^2\,y^2 + \tan(x)^2}{(y\tan(x)-1)^2}\right)^{(1/4)}\right) = 0$$

reversing the transformation used

```
> gitr(TR1);
```

$$\{x = t,\ y = \tan(\mathrm{u}(t))\}$$

and changing variables now in the *solution* just obtained we arrive at the solution we are looking for the original problem, ode_{81}:

```
> ch(%,a4);
```

$$\_C1+\frac{\tan(x)}{\left(\left(\frac{1}{\tan(x)}+\frac{\tan(x)^2}{\frac{\tan(x)^4\tan(y)}{1+\tan(x)^2}-\frac{\tan(x)^3}{1+\tan(x)^2}}\right)^2+1\right)^{(1/4)}}-\frac{1}{2}(\tan(y)+\tan(x))$$

$$\left(-2+\operatorname{hypergeom}\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2}\right],-\frac{(\tan(y)+\tan(x))^2}{(\tan(y)\tan(x)-1)^2}\right)\left(\frac{\tan(y)^2+1+\tan(x)^2\tan(y)^2+\tan(x)^2}{(\tan(y)\tan(x)-1)^2}\right)^{(1/4)}\right)$$

$$\Bigg/\left((\tan(y)\tan(x)-1)\left(\frac{\tan(y)^2+1+\tan(x)^2\tan(y)^2+\tan(x)^2}{(\tan(y)\tan(x)-1)^2}\right)^{(1/4)}\right)=0$$

First order ODEs of higher degree

For first order ODEs of higher degree, for instance Kamke's examples 395, 460, 537, the techinque just explained unfortunately does not work. The reason is that, to map into a Riccati or Abel type ODE one would need to isolate *y'* anyway, thus introducing radicals all around, making the approach unpractical.

Nonetheless, the idea just exposed is still usable if:

- Instead of a simple transformation of the form $\{x=t, y=\mathrm{P}(\mathrm{u}(t))\}$, we consider a transformation of the form

$$\{x=t, y=\mathrm{P}(t, \mathrm{u}(t))\}$$

- Instead of mapping into a Riccati or Abel type ODE we consider mapping into a ***d'Alembert type ODE***

$$\mathrm{y}(x)=x\,\mathrm{F}(y')+\mathrm{G}(y')$$

That is, an equation non-linear in *y'*, but linear in *x* and *y*. The first experiments showed that this approach - although based on mapping a ***first order ODE*** into a ***first order non-linearPDE*** - it works fine for solving these high-degree remaining *difficult* examples.