Rapid computation of Bernoulli and related numbers.

1 Abstract.

Bernoulli numbers and similar arithmetic objects have long been of interest in mathematics. Historically, people have been interested in different recursion formulae that can be derived for the Bernoulli numbers, and the use of these recursion formulae for the calculation of Bernoulli numbers. Some of these methods, which in the past have only been of theoretical interest, are now practical with the availability of high-powered computation.

This poster explores some of these techniques of deriving new recursion formulae, and expands upon these methods. The main technique that is explored is that of "*multisectioning*". Typically, the calculation of a Bernoulli number requires the calculation of all previous Bernoulli numbers. The method of multisectioning is such that only a fraction of the previous Bernoulli numbers are needed. In exchange, a more complicated recursion formula, called a "*lacunary recursion formula*", must be derived and used.

2 History

Bernoulli numbers were first introduced by Jacques Bernoulli (1654-1705), in the second part of his treatise published in 1713, *Ars conjectandi* ("Art of Conjecturing"). At the time, Bernoulli numbers were used for writing the infinite series expansions of hyperbolic and trigonometric functions [2].

Van den Berg was the first to discuss finding recurrence formulae for the Bernoulli numbers with arbitrary sized gaps (1881) [4]. Ramanujan showed how gaps of size 7 could be found, and explicitly wrote out the recursion for gaps of size 6 [1, 4, 5]. Lehmer in 1934 extended these methods to Euler numbers, Genocchi numbers, and Lucas numbers (1934) [4], and calculated the 196-th Bernoulli number. These methods to find new recurrences for Bernoulli numbers, and numbers related to the Bernoulli numbers were automated in Maple, and discussed in a much more general setting for my thesis [3].

3 Poster

We begin with two definitions.

Definition 3.1 (Poly-exponential function.) *Let* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *be constants and* $p_1(x), \ldots, p_n(x) \in \mathbb{C}[x]$ *be polynomials. Then*

$$\sum_{i=1}^{n} p_i(x) e^{\lambda_i x},$$

is a "poly-exponential function"*. Denote the set of all such functions by* $\mathcal{P}$*.*

Definition 3.2 (Rational poly-exponential function.) *Let $s(x)$, $t(x) \in \mathcal{P}$ and $t(x) \neq 0$. Then*

$$\frac{s(x)}{t(x)},$$

is a "rational poly-exponential function"*. Denote the set of all such functions by $\mathcal{R}$.*

We see that there are numerous examples of rational poly-exponential functions that are of interest in mathematics. Two common examples are the Bernoulli numbers and Euler numbers, (with an exponential generating functions of $\frac{x}{e^x-1}$ and $\frac{2}{e^x+e^{-x}}$ respectively).

Definition 3.3 (Multisectioning.) *Let $f(x)$ be a function acting on a subset of $\mathbb{C}$. Define $f_m^q(x) = \frac{1}{m}\sum_{i=0}^{m-1} \omega_m^{-iq}\ f(\omega_m^i x)$, where $\omega_m = e^{\frac{2\pi i}{m}}$.*

Multisectioning is used to help prove the following theorem.

Theorem 3.1 *Given a function $f(x) \in \mathcal{R}$, $m, q \in \mathbb{Z}$, $0 \leq q < m$, a recursion formula can be found for the $mi + q$-th coefficient of the exponential generating function of $f(x)$ that depends only on the $mj + q$-th coefficient, for $j < i$, and two lacunary recurrence relations.*

This is the main theorem that allows the calculation of Bernoulli and related numbers to be done quickly. This is because only $\frac{1}{m} - th$ of the numbers need to be computed at the intermediate steps. The main emphasis for this poster will be the calculation of these lacunary recurrence relations. (These calculations become increasingly more difficult for large m.)

References

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[5] S Ramanujan *Some properties of Bernoulli's numbers* Indian Mathematical Journal, December, 1911