The Sensitivity of Traveltime Inversion for an Anisotropic Parameter in Seismology

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Abstract: Seismic anisotropy is the variation of the velocity of a signal as a function of the signal propagation direction through a medium. It plays an important yet subtle role in exploration seismology. The main measurement of exploration seismology is traveltime. This is the time it takes for a signal to propagate from a source to a receiver. In this paper we study one method for calculating anisotropy from a single traveltime measurement. In particular, we study the sensitivity of anisotropy to measurement errors in the traveltime. Maple was used to derive exact formulae for the traveltime and anisotropy. Using these formulae we perform a symbolic analysis of the sensitivity. We determine that weak anisotropy is inherently sensitive to measurement errors in the traveltime but we suggest certain modifications to the parameters to reduce other contributions to the sensitivity. The formulae obtained should prove helpful to geophysicists in pre-survey planning and decision making.

Keywords: seismic anisotropy, traveltime inversion, Thomsen's γ parameter, elasticity, symbolic analysis.

Introduction

In seismology an *isotropic* medium denotes a material for which the velocity of a signal is independent of the signal propagation direction through the material. A material which is not isotropic is called *anisotropic*. For example, shales are anisotropic. They are composed of thin laminae. A sound wave travels faster in the parallel direction of the laminae than in the perpendicular direction.

The assumption of isotropic media has been widely used in exploration seismology for two main reasons. Firstly, most media are weakly anisotropic. Secondly, the equipment for data acquisition is often unable to detect consequences of weak anisotropy. Furthermore, the assumption of isotropy makes the mathematics involved significantly simpler. Nevertheless, today it is more difficult to locate potential oil or gas reserves and the equipment that is used to measure the traveltime for sound to travel between a source and a receiver can now measure the time to within one or two milliseconds. Thus consideration of anisotropy becomes important for two fundamental reasons, namely, to determine lithology and to provide proper imaging. Lithology entails a knowledge about the characteristics of the material, for example, its porosity, fracturing, fluid/gas content, etc. Imaging provides a picture of a subsurface thus defining the spatial location of a given feature, for instance, a fault or a fold.

Most seismic data is acquired with both source and receiver located on the surface. However, one of the best ways to study anisotropic properties of materials is vertical seismic profiling (VSP) where the source is placed on the surface while the receivers are placed in a wellbore. This is the approach assumed in this paper. We derive an exact expression for an anisotropy related to SH waves (shear waves with horizontal particle displacement) from VSP traveltime measurements. This approach can also be applied to other wave types which exhibit an approximately elliptical velocity dependence.

Consider the following model in which an anisotropic layer is buried under an isotropic surface layer.

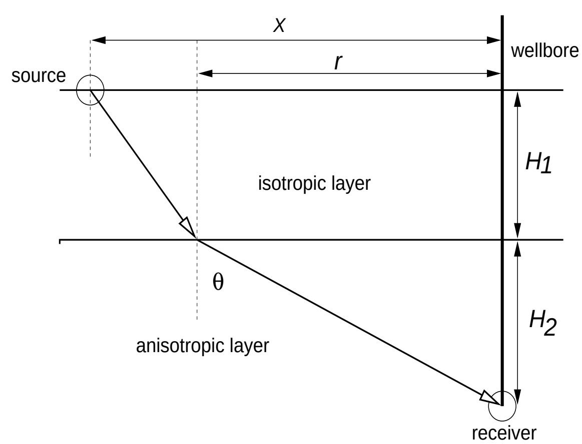

Figure 1: Two-layer isotropic/anisotropic model

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‡The Sponsors of The Geomechanics Project: PanCanadian Petroleum Ltd., Petro-Canada Oil and Gas, Talisman Energy Inc., and Western Atlas International Inc.

The interface between the surface and the buried layer is assumed to be horizontal (a reasonable assumption for sedimentary basins which are of interest in petroleum exploration) and the layers are assumed to be of uniform nature. From the wellbore information we know H_1 the depth of the interface. The receiver is placed in the second layer at depth H_2 below the interface. The source is located on the surface with lateral source receiver offset denoted by X. The vertical velocity of both layers is assumed to be known from a measurement taken with the source directly above the receiver. The vertical velocity in the isotropic surface layer is denoted V_1. It is a constant independent of propagation direction. The vertical velocity of the buried layer is denoted β. The horizontal velocity α is assumed to be $\beta\sqrt{1+2\gamma}$. For intermediate phase angles ϕ the velocity is assumed to have an elliptical dependence given by

$$v(\phi) = \beta\,\sqrt{1 + 2\,\gamma \sin(\phi)^2}. \tag{1}$$

The parameter γ (Thomsen [3]) characterizes the anisotropy. It may be positive or negative. The value $\gamma = 0$ means the medium is isotropic.

Since the layers are assumed to be homogeneous the signal propagates in straight lines through them. At the interface, however, the signal will refract. The location of the refraction point, denoted as r in Figure 1, is of interest for imaging and locating features in the subsurface.

The traveltime, the time for the signal to propagate from the source to the receiver, is given by the following formula

$$T = \frac{\sqrt{(X-r)^2 + H_1^2}}{V_1} + \frac{\sqrt{r^2 + H_2^2}}{V(\theta)} \tag{2}$$

where $V(\theta)$ is the velocity through the anisotropic layer and θ is the called the group angle or the propagation angle. In this paper we use the general equations proposed by Thomsen [3]. Although equation (1) involves phase velocity, v, the signal between a point source and a point receiver propagates with group velocity V given by

$$\mathrm{V}(\theta(\phi)) = \sqrt{v(\phi)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}\phi}\,v(\phi)\right)^2} \tag{3}$$

where the relation between phase angle, ϕ, and group angle, θ, is

$$\tan(\theta(\phi)) = \frac{\tan(\phi) + \dfrac{\frac{\mathrm{d}}{\mathrm{d}\phi}\,v(\phi)}{v(\phi)}}{1 - \dfrac{\tan(\phi)\,(\frac{\mathrm{d}}{\mathrm{d}\phi}\,v(\phi))}{v(\phi)}}. \tag{4}$$

The approach taken to calculate γ involves measurement of the traveltime, T. Since H_1, H_2, V_1, β, and X will be known, the remaining unknowns in equation (2) will be γ and r. The mathematical calculation yielding the point r is based on Fermat's principle of stationary time. We assume that the signal trajectory minimizes the traveltime T of the signal from the source to the receiver.

Slawinski et al. [1] used the linear approximation $\beta(1 + \gamma \sin^2\phi)$ of equation (1). For the model depicted in Figure 1 they obtained

$$T = \frac{\sqrt{(X-r)^2 + H_1^2}}{V_1} + \frac{\sqrt{r^2 + H_2^2}}{\beta(1 + \gamma\frac{r^2}{r^2+H_2^2})}.$$

Inverting this formula exactly for γ to obtain γ as a function of the traveltime, one obtains

$$\gamma = \frac{V_1\sqrt{r^2 + H_2^2}^3}{\beta r^2 (TV_1 - \sqrt{(X-r)^2 + H_1^2}} - \frac{r^2 + H_2^2}{r^2}.$$

The value of γ may now be estimated from a single travel time measurement.

From numerical experiments Slawinski et al. [1] observed that the value of γ calculated in this way is very sensitive to errors in T. At the Pacific Institute of the Mathematical Sciences industrial workshop in August 1997, finding a more robust technique for measuring γ was proposed as one of six industrial problems by Michael Slawinski. At the workshop, Lamoureux et al. [2] presented initial results confirming the sensitivity of γ and showing that the horizontal velocity is less sensitive than γ to measurement errors in T.

In this paper we do not use a linear approximation for $v(\phi)$. Instead we obtain exact expressions for $V(\theta)$, T, and γ. This allows us to study with analytical methods the sensitivity of γ on T. We will show that small γ, i.e. weak anisotropy, is inherently sensitive to T. We do this by a symbolic analysis of a measure for the sensitivity. The analysis also reveals how one can design a near optimal measurement.

Understanding the sensitivity of γ will be important in seismic exploration where substantial economic and technical decisions are based on imperfect information and hence the knowledge of errors is critical in the decision process, thus emphasizing the importance of presurvey planning. Thus before one performs a seismic experiment one can now calculate the sensitivity of the actual experiment.

The paper is organized as follows. First we present the results. The detailed Maple calculations, are presented in an Appendix. The role Maple played in this work was to find the exact formulae and to facilitate the manipulation and simplification of moderately large formulae in the symbolic analysis. The numerical calculations and graphs were helpful in understanding the meaning of the formulae.

The interested reader is referred to the book by Helbig [4] which describes many interesting and important details from both the theoretical and practical viewpoints associated with isotropy in seismology.

Results

The exact expression for the group velocity, V, as a function of group angle, θ, which directly describes the velocity and trajectory of a signal between a point source and a point receiver is given by the following formula obtained by Maple

$$V(\theta) = \beta \sqrt{\frac{1 + 2\,\gamma}{2\,\gamma \cos(\theta)^2 + 1}}. \tag{5}$$

Substituting for $V(\theta)$ into equation (2) for the traveltime, replacing $\cos(\theta)$ by $H_2/\sqrt{r^2 + H_2^2}$ yields the following formula for the traveltime

$$T = \frac{\sqrt{(X - r)^2 + {H_1}^2}}{V_1} + \frac{\sqrt{{H_2}^2 + \dfrac{r^2}{1 + 2\,\gamma}}}{\beta}. \tag{6}$$

Solving for γ yields the following formula which we can use to estimate γ from a single measurement taken for the traveltime T.

$$\gamma = \frac{1}{2} \frac{r^2}{(-T + \dfrac{\sqrt{(X - r)^2 + {H_1}^2}}{V_1})^2\,\beta^2 - {H_2}^2} - \frac{1}{2}. \tag{7}$$

It was a surprise to us to find that these exact results are not more complicated than the formulae obtained in [1] which used a linear approximation for $v(\phi)$.

Sensitivity of γ to Errors in T

We illustrate the sensitivity of γ on T by a simple numerical experiment. Consider the following values: $H_1 = 355m, H_2 = 1045m,\ X = 590m, V_1 = 1030m/s, \beta = 1609m/s$ and $\gamma = 0.096$ which describe a weakly anisotropic subsurface layer typical for geophysical media. The refraction point and the traveltime computed using these values are $r = 503.5027691m$ and $T = 1.064648399s$ respectively.

Now suppose the there is a 1% error in the measured traveltime, thus $T = 1.075294883s$. What effect will this have on the value of γ calculated using equation (7)? Solving for γ using this value of T yields $\gamma = 0.0008084735$ which has no agreement with $\gamma = 0.096$.

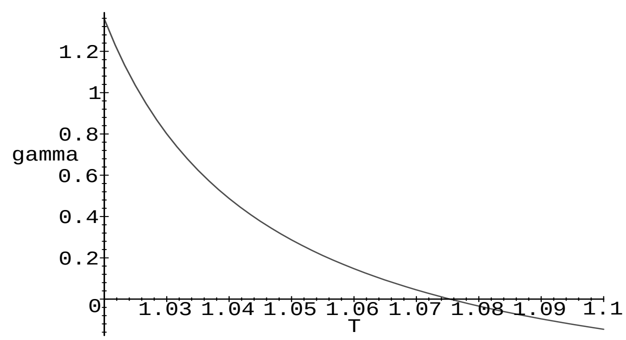

Figure 2: Sensitivity of γ on T

Figure 2 shows γ as a function of T. It illustrates the sensitivity of γ on T.

Observe that a 2% error in T yields a negative value for γ which would imply the velocity decreases with the angle of propagation instead of increasing. Such an error in γ would imply a significant difference in the lithology.

We would like to quantitatively evaluate the accuracy of γ for a given accuracy of T. The following function measures the sensitivity of a function $F(T)$ to change in T.

$$E(F(T)) = \frac{F'(T)}{F(T)} T$$

If $|E| \gg 1$ then $F(T)$ is sensitive to change in T and if $|E| \ll 1$ then $F(T)$ is insensitive to change in T. Economists call $E(F(T))$ the *elasticity* of $F(T)$. The factor $F'(T)/F(T)$ captures the relative change in $F(T)$. For our application we obtain

$$E(\gamma(T)) = \frac{r^2 A \beta^2 T}{2(A^2\beta^2 - H_2^2)^2(\frac{r^2}{A^2\beta^2 - H_2^2} - 1)} \tag{8}$$

where

$$A = -T + \frac{\sqrt{(X - r)^2 + H_1^2}}{V_1}.$$

Evaluating this formula using the previous data for H_1, H_2, X, V_1, β, γ, r and T yields $E(\gamma(T)) = -114.23$. This implies a hundredfold amplification of an error in T.

We supposed that the cause of this sensitivity might be due to the offset, $X = 590m$, being relatively small compared to the measured depth, $H_1 + H_2 = 1400m$. If X were larger, the signal should travel through the buried layer closer to the horizontal direction.

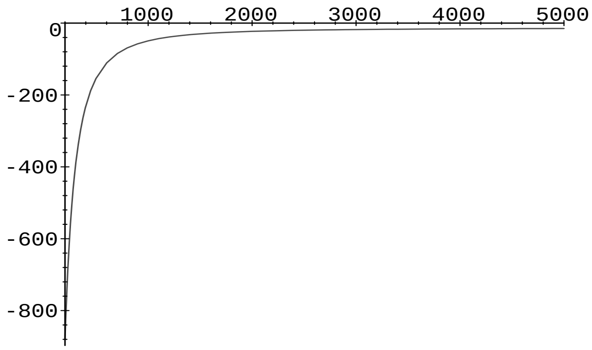

Figure 3: Elasticity of $\gamma(T)$ versus X

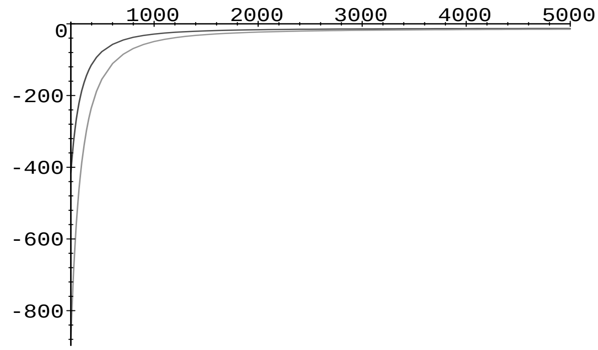

Figure 4: Elasticity for one- and two-layer models

Figure 3 plots sensitivity as a function of X with all other parameters the same. Observe that the magnitude of the sensitivity increases to a singularity at $X = 0$ which corresponds to a vertical signal trajectory. It is not possible to measure anisotropy in such a case. Observe also that for large X the graphs show that γ remains sensitive. At $X = 1000m$ the sensitivity is -49 and at $X = 5000m$ it is still -15.

On the basis of this single numerical experiment one may propose that one should make X large. However, there are two important considerations which limit X. Firstly, at a certain offset X one reaches a critical angle beyond which the signal will entirely reflect off the interface. Secondly, large offsets imply large distances and the signal may be too weak to be accurately detected.

The obvious solution to the problem would be to measure the traveltime horizontally through the subsurface layer by placing the source in an adjacent well. This approach, called cross-well tomography, is more expensive and is not always possible.

Figure 4 shows the sensitivity as a function of X where we assume only one anisotropic layer i.e. $H_1 = 0m$, effectively what happens when $H_1 \ll H_2$. In this case we do not need to consider the refraction point r, i.e. the signal travels in a straight line from the source to the receiver. Thus the only unknown is γ. The graph also includes the sensitivity from the two-layer model shown in Figure 3. One observes that the magnitude of the sensitivity when $H_1 = 0$ is always less than when $H_1 > 0$.

Symbolic Analysis of Sensitivity

We seek to analyze the sensitivity of γ with respect to errors in T for both the two layer and one layer models. Let us begin with the case of a single anisotropic layer for which the analysis is relatively simple. Here the traveltime is given by

$$T = \frac{\sqrt{{H_2}^2 + \frac{X^2}{1+2\gamma}}}{\beta}, \tag{9}$$

γ is given by

$$\gamma = \frac{1}{2}\,\frac{X^2}{T^2\,\beta^2 - {H_2}^2} - \frac{1}{2}, \tag{10}$$

and the elasticity is given by

$$E = 2\,\frac{\beta^2\,T^2\,X^2}{(T^2\,\beta^2 - {H_2}^2 - X^2)\,(T^2\,\beta^2 - {H_2}^2)}. \tag{11}$$

To interpret this as a function of γ and the other parameters, instead of T, we substitute for T from equation (9) to obtain

$$E = -\frac{(1+2\,\gamma)\,({H_2}^2 + 2\,\gamma\,{H_2}^2 + X^2)}{X^2\,\gamma}. \tag{12}$$

Now we assume weak anisotropy. We replace $1+2\gamma$ with 1 to obtain

$$E = -\frac{{H_2}^2 + X^2}{X^2\,\gamma}. \tag{13}$$

Thus, the sensitivity, as measured by elasticity, is inversely proportional to γ which implies that small values of γ will be difficult to detect from the traveltime.

The factor of X^2 in the denominator confirms the singularity, i.e. $X = 0$ does not allow the determination of γ. Also if $H_2 \gg X$ then the elasticity will multiplied by $(H_2/X)^2$. Thus we conclude the following. Firstly, one should choose X at least as large as H_2. Secondly, one cannot remove the γ^{-1} factor from the elasticity.

The Two-Layer Case

For the two layer case we proceed in the same manner. We replace T in equation (8) by T in equation (6). We again make the weak anisotropy assumption replacing $1 + 2\gamma$ by 1. After simplification we obtain

$$E = \frac{\sqrt{r^2 + H_2^2}}{r^2 \gamma V_1}(V_1\sqrt{r^2 + H_2^2} + \beta\sqrt{(X - r)^2 + H_1^2}). \tag{14}$$

Again, we see that the elasticity is multiplied by γ^{-1}. Thus in our numerical experiment, where γ was 0.096 and the elasticity was -114.23, a factor of 10.4 is due to γ being small.

In equation (14) we see that other contributions to large elasticity will come from two sources, firstly, the relative sizes of the velocities V_1 and β due to the factor of V_1 in the denominator, and secondly, from the relative sizes of the distances H_1, H_2, X and r due to the factor of r^2 in the denominator.

We begin by considering the cases $V_1 \gg \beta$, $V_1 \ll \beta$ and $V_1 = \beta$. For the case $V_1 = \beta$ we obtain

$$E = \frac{\sqrt{r^2 + H_2^2}}{\gamma r^2}(\sqrt{r^2 + H_2^2} + \sqrt{(X - r)^2 + H_1^2}).$$

This is large when r is small compared with any of X, H_1, H_2. These cases are examined later.

For the case $V_1 \gg \beta$ we obtain

$$E = \frac{r^2 + H_2^2}{\gamma r^2}$$

which is large if $H_2 \gg r$. This case is examined later.

For the case $\beta \gg V_1$ we obtain

$$E = \frac{\beta\sqrt{r^2 + H_2^2}\sqrt{(X - r)^2 + H_1^2}}{V_1 \gamma r^2}$$

which multiplies the sensitivity by the factor β/V_1. This is also large if $H_2 \gg r$, $H_1 \gg r$, or $X \gg r$. These cases are examined later.

If $\beta \gg V_1$ then the traveltime in the anisotropic layer will be small (unless $H_1 \ll H_2$, a case which is examined later) hence why γ is sensitive. However, in practice, β will typically not differ from V_1 by more than a factor of 2.

We now consider the cases $H_2 \gg X$ and $H_1 \gg X$. The cases $H_2 \gg r$ and $H_1 \gg r$ are included since $0 < r < X$. We also assume β is comparable in magnitude to V_1. For $H_2 \gg X$ the elasticity simplifies to

$$E = -\frac{H_2^2 + H_2\sqrt{(X - r)^2 + H_1^2}}{r^2 \gamma}.$$

Since the two quantities in the numerator are both positive then the elasticity is multiplied by a factor of at least $(H_2/r)^2$ which is very large.

For the case $H_1 \gg X$ and $H_2 \not\gg X$ (we have already considered the case $H_2 \gg X$), the elasticity simplifies to

$$E = -\frac{r^2 + H_2^2 + H_1\sqrt{r^2 + H_2^2}}{r^2 \gamma}$$

which introduces a factor of at least H_1/r into the sensitivity. To avoid these factors in the sensitivity we need to ensure that r is not significantly smaller than H_2 and H_1. This will usually be the case if X is chosen to be at least as large as H_1 and H_2, or, simply $X \geq H_1 + H_2$.

We now consider the cases $H_1 \gg H_2$ and $H_2 \gg H_1$. The case $H_2 \gg H_1$ reduces to the previously discussed one layer case. If $H_1 \gg H_2$, that is, the receiver is placed just below the interface, then r will normally be small because the point of refraction will be close to the receiver. If H_2 and r are both small, the elasticity becomes

$$E = -\frac{\sqrt{X^2 + H_1^2}}{r\gamma},$$

which shows that the elasticity is proportional to X/r or H_1/r. This sensitivity can be resolved by ensuring H_2 and X is not small compared with H_1.

If $H_1 \gg H_2$ and the point of refraction is not close to the receiver then the signal travels near horizontally through the subsurface layer. This happens if $\beta \gg V_1$ because the signal travels a larger distance through the faster medium.

The first conclusion from these results is that weak anisotropy is inherently sensitive to the traveltime because of the factor of γ^{-1} in the elasticity. The second conclusion is that, except for the case $\beta \gg V_1$, the other contributions to the sensitivity can be eliminated by appropriate choices of H_2 and X. In particular ensuring that $H_2 \geq H_1$ and $X \geq H_1 + H_2$ will be sufficient for many practical purposes.

Consider therefore modifying our original numerical experiment to use $X = 1400m = H_1 + H_2$ instead of $X = 590m$. We determine that $r = 1230.9m$ and the traveltime is $1.3372m$ and we obtain the reduced elasticity of -32.308 compared with -114.23, a reduction of approximately a factor of 3.5. Keeping $X = 1400m$, consider reducing the size of H_2 to be the same as H_1

to force the signal to travel more horizontally in the buried layer. We determine that $r = 1162.39m$ and $T = 1.1122s$ and we obtain $E = -22.001$, a reduction of a factor of 5.2. Since a factor of 10.4 in E is due to $\gamma = 0.096$ we are close to the best we can do without making X too large.

Conclusions

We determined an exact formula for the traveltime T for SH waves for the case of an isotropic surface layer above an anisotropic layer. From this formula we obtained an exact formula for Thomsen's anisotropy parameter γ. Because we were able to obtain exact formulae we are able to symbolically analyze what can cause γ to be sensitive to measurement error of T.

We determined that one contribution to the sensitivity is γ^{-1}. Thus computing γ from the traveltime T is inherently sensitive to errors in traveltime measurements when γ is small.

A second contribution occurs when the horizontal offset X is too small relative to H_1 or H_2, implying near-vertical signal trajectories. A third contribution occurs when $H_1 \gg H_2$, the receiver is not sufficiently deep in the buried layer resulting in insufficient traveltime in the buried layer. One can avoid theses third contributions by an appropriate choice of X and H_2 which ensures that a large proportion of the traveltime is spent traveling horizontally through the anisotropic layer.

A fourth contribution occurs when $\beta \gg V_1$, the vertical velocity in the subsurface layer is much faster than in the surface layer. This will not commonly occur in practice.

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Biographies

Michael Monagan is assistant professor in mathematics and statistics at Simon Fraser University. A developer of the Maple system, he is interested in applications of symbolic computation in the physical sciences and the use of systems such as Maple to prepare numerical codes; he is an author of Maple facilities to generate Fortran and C code. His other research interests include simplification of formulae, polynomial factorization, greatest common divisors, and resultants, and non-linear dynamical systems.

Michael A. Slawinski is industrial research professor in the Department of Mechanical Engineering at The University of Calgary where he is leading The Geomechanics Project which is sponsored by four industrial partners. The purpose of the project is to perform rigorous theoretical research leading to industrial applications. He did his Ph.D. in geophysics (seismology). Prior to this appointment he worked at Petro-Canada, PanCanadian and Chevron as an exploration geophysicist. His present research interests include lithology determination and imaging using seismic wave propagation.

Appendix

This appendix contains the Maple commands used to produce the results previously shown in the paper. Note, the two plots produced by Maple are not reproduced here.

```
> alias(gamma=Gamma): assume(beta>0, H1>0, H2>0, V1>0, V2>0);
```

This is the basic formula for the traveltime T.

```
> e1 := T = sqrt( (X-r)^2 + H1^2 )/V1 + sqrt(r^2+H2^2) / V2;
```

$$e1 := T = \frac{\sqrt{X^2 - 2\,X\,r + r^2 + H1^2}}{V1} + \frac{\sqrt{r^2 + H2^2}}{V2}$$

```
> e2 := sqrt( v(phi)^2 + diff(v(phi),phi)^2):
> e3 := (tan(phi)+1/v(phi)*diff(v(phi),phi)) / (1-tan(phi)/v(phi)*diff(v(phi),phi)):
> e4 := v(phi) = beta*sqrt(1+2*gamma*sin(phi)^2);
```

$$e4 := \mathrm{v}(\phi) = \beta\,\sqrt{1 + 2\,\gamma\sin(\phi)^2}$$

```
> e5 := subs(e4,e2):
> e5 := simplify(e5,sqrt):
> e5 := simplify(e5,{sin(phi)^2+cos(phi)^2=1},{cos(phi)});
```

$$e5 := \beta\,\sqrt{\frac{1 + 4\,\gamma\sin(\phi)^2 + 4\,\gamma^2\sin(\phi)^2}{1 + 2\,\gamma\sin(\phi)^2}}$$

```
> e6 := subs(e4,e3):
> e6 := normal(e6);
```

$$e6 := \frac{\tan(\phi) + 2\tan(\phi)\,\gamma\sin(\phi)^2 + 2\,\gamma\sin(\phi)\cos(\phi)}{1 + 2\,\gamma\sin(\phi)^2 - 2\tan(\phi)\,\gamma\sin(\phi)\cos(\phi)}$$

```
> convert(%,sincos):
> e6 := simplify(%);
```

$$e6 := \frac{\sin(\phi)\,(1 + 2\,\gamma)}{\cos(\phi)}$$

```
> e7 := phi = arctan( tan(theta)/(1+2*gamma)):
```

The formula for $V(\theta)$ is obtained below in `e8`. Prior to simplification it is a moderately large formula. In the following commands, Maple is used to simplify this formula into a more useful form. It was this simplification that permitted us to find simple analytical expressions for T and γ, and in turn, a relatively simple analytical expression for $E(\gamma(T))$.

```
> e8 := subs(e7,e5):
> e8 := simplify(e8):
> convert(e8,sincos):
> e8 := simplify(%,{sin(theta)^2+cos(theta)^2=1},{sin(theta)});
```

$$e8 := \beta\,\sqrt{\frac{1 + 2\,\gamma}{2\,\gamma\cos(\theta)^2 + 1}}$$

```
> EQN := subs( V2=e8, cos(theta)=H2/sqrt(r^2+H2^2), e1 );
```

$$EQN := T = \frac{\sqrt{X^2 - 2\,X\,r + r^2 + HI^2}}{VI} + \frac{\sqrt{r^2 + H2^2}}{\beta\sqrt{\dfrac{1 + 2\,\gamma}{2\,\dfrac{\gamma\,H2^2}{r^2 + H2^2} + 1}}}$$

We want to combine the two square roots on the right. Maple V Release 4 is not able to do this nicely with `combine(%,radical,symbolic)`. We first simplify what's under the combined square root – we use the partial fraction form which turns out to be the simplest.

```
> convert((r^2+H2^2)/(1+2*gamma)*(1+2*gamma*H2^2/(r^2+H2^2)), parfrac, gamma):
> EQN := T = ((X-r)^2+H1^2)^(1/2)/V1 + (%)^(1/2)/beta: EQN;
```

$$T = \frac{\sqrt{(X - r)^2 + HI^2}}{VI} + \frac{\sqrt{H2^2 + \dfrac{r^2}{1 + 2\,\gamma}}}{\beta}$$

The above expression, therefore, gives us the traveltime T. A formula for calculating the anisotropy parameter γ can be obtained by simply solving for γ.

```
> sol := solve(EQN,gamma): sol;
```

$$-\frac{1}{2}(-r^2\,VI^2 - H2^2\,VI^2 + \beta^2\,T^2\,VI^2 - 2\,\beta^2\,T\,VI\,\sqrt{X^2 - 2\,X\,r + r^2 + HI^2} + \beta^2\,X^2 - 2\,\beta^2\,X\,r + \beta^2\,r^2 + \beta^2\,HI^2)$$
$$\Big/(-H2^2\,VI^2 + \beta^2\,T^2\,VI^2 - 2\,\beta^2\,T\,VI\,\sqrt{X^2 - 2\,X\,r + r^2 + HI^2} + \beta^2\,X^2 - 2\,\beta^2\,X\,r + \beta^2\,r^2 + \beta^2\,HI^2)$$

The result returned by `solve` can be simplified considerably, infact, to the formula below. We first did this by hand as Maple is unable to do the simplification. Later we realized that if one solves for γ in the obvious way by hand one would obtain the simplified formula directly. The Maple library routine `isolate` mimics what one would do by hand. Maple is, however, able to verify that the two formulae do infact agree.

```
> Ans := readlib(isolate)(EQN,gamma): Ans;
```

$$\gamma = \frac{1}{2}\,\frac{r^2}{(-T + \dfrac{\sqrt{(X - r)^2 + HI^2}}{VI})^2\,\beta^2 - H2^2} - \frac{1}{2}$$

```
> simplify(rhs(Ans)-sol);
```

$$0$$

Start with $\gamma = 0.096$ and compute r and T

```
> subs( H1=355, H2=1045, X=590, V1=1030, beta=1609, gamma=0.096, EQN ):
> R := fsolve(diff(%,r),r);
```

$$R := 503.5027691$$

```
> t := subs( r=R, rhs(%%) );
```

$$t := 1.064648399$$

Now assume we make an error of $+1\%$ in measuring T, i.e

```
> subs( H1=355, H2=1045, X=590, V1=1030, beta=1609, T=t+0.01*t, rhs(Ans) ):
> r = fsolve(diff(%,r),r=1..590);
```

$$r = 489.9208918$$

```
> gamma = subs( %, %% );
```

$$\gamma = .0008084735$$

Now we graph γ as a function of T to visualize how γ depends on T.

```
> G := proc(t) local R;
  subs( H1=355, H2=1045, X=590, V1=1030, beta=1609, T=t, rhs(Ans) );
  R := fsolve(diff(%,r),r,r=1..590);
  subs(r=R,%%)
  end:
```

```
> plot( G, 1.02 .. 1.10, labels=[T,gamma] );
```

Define the elasticity of $\mathrm{E}(\mathrm{F}(T)) = \frac{\mathrm{F}'(T)}{\mathrm{F}(T)}T$.

```
> E := diff(rhs(Ans),T)/rhs(Ans)*T;
```

$$E := \frac{r^2\,(-T + \dfrac{\sqrt{(X-r)^2 + \mathit{H1}^2}}{\mathit{V1}})\,\beta^2\,T}{((-T + \dfrac{\sqrt{(X-r)^2 + \mathit{H1}^2}}{\mathit{V1}})^2\,\beta^2 - \mathit{H2}^2)^2\left(\dfrac{1}{2}\,\dfrac{r^2}{(-T + \dfrac{\sqrt{(X-r)^2 + \mathit{H1}^2}}{\mathit{V1}})^2\,\beta^2 - \mathit{H2}^2} - \dfrac{1}{2}\right)}$$

```
> subs( r=R, H1=355, H2=1045, X=590, V1=1030, beta=1609, T=t, E );
```

$$-114.2341664$$

Now graph the elasticity as a function or T for the given values of *H1, H2, V1*, β.

```
> E := subs( H1=355, H2=1045, V1=1030, beta=1609, E ):
```

```
> F := subs( H1=355, H2=1045, V1=1030, beta=1609, gamma=0.096, rhs(EQN) ):
```

```
> Fr := diff(F,r):
```

```
> ME := proc(x) local R,t;
  R := fsolve(subs(X=x,Fr),r,r=1..x);
  if not type([R],[numeric]) then ERROR(`r Failure at X = `,x) fi;
  t := evalf(subs(X=x,r=R,F));  evalf(subs(X=x,r=R,T=t,E));
  end:
```

```
> ME(1000); ME(5000);
```

$$-49.36446047$$

$$-15.11159567$$

```
> plot(ME,200..5000);
```

We analyse the elasticity for both models to see under what conditions it is large. Let us begin with the simple case of a single anisotropic layer.

```
> eqn := subs( H1=0, r=X, EQN );
```

$$\mathit{eqn} := T = \frac{\sqrt{\mathit{H2}^2 + \dfrac{X^2}{1 + 2\,\gamma}}}{\beta}$$

```
> ans := isolate( eqn, gamma ):
> e := diff(rhs(ans),T)/rhs(ans)*T;
```

$$e := -\frac{X^2\,T^2\,\beta^2}{(T^2\,\beta^2 - H2^2)^2\,(\frac{1}{2}\,\frac{X^2}{T^2\,\beta^2 - H2^2} - \frac{1}{2})}$$

We substitute for T obtaining the elasticity as a function of γ rather than T and consider the case where γ is small, hence $1 + 2\,\gamma$ is approximately 1.

```
> subs( eqn, e );
```

$$-\frac{(1+2\,\gamma)^2\,(H2^2 + \frac{X^2}{1+2\,\gamma})}{X^2\,\gamma}$$

```
> algsubs(1+2*gamma=1, %);
```

$$-\frac{X^2 + H2^2}{X^2\,\gamma}$$

Now we do this analysis for the two-layer model.

```
> E := diff(rhs(Ans),T)/rhs(Ans)*T:
> subs(EQN, E);
```

$$-\frac{(1+2\,\gamma)^2\,\sqrt{H2^2 + \frac{r^2}{1+2\,\gamma}}\,\beta\,\left(\frac{\sqrt{(X-r)^2 + H1^2}}{V1} + \frac{\sqrt{H2^2 + \frac{r^2}{1+2\,\gamma}}}{\beta}\right)}{r^2\,\gamma}$$

```
> subs( 1+2*gamma=1, % );
```

$$-\frac{\sqrt{r^2 + H2^2}\,\beta\,(\frac{\sqrt{(X-r)^2 + H1^2}}{V1} + \frac{\sqrt{r^2 + H2^2}}{\beta})}{r^2\,\gamma}$$

```
> normal(%,expanded):
> combine(%,radical,symbolic):
> collect(%,V1,factor);
```

$$-\frac{r^2 + H2^2}{r^2\,\gamma} - \frac{\beta\,\sqrt{(r^2 + H2^2)\,(X^2 - 2\,X\,r + r^2 + H1^2)}}{V1\,r^2\,\gamma}$$

Evaluating the sensitivity at $X = 1400$ we obtain

```
> subs( H1=355, H2=1045, X=1400, V1=1030, beta=1609, gamma=0.096, EQN ):
> R := fsolve( diff(rhs(%),r), r );
```

$$R := 1230.910585$$

```
> t := subs(r=R,rhs(%%));
```

$$t := 1.337162700$$

```
> subs( H1=355, H2=1045, X=1400, V1=1030, beta=1609, r=R, T=t, E );
```

$$-32.30807743$$