# Abel ODEs:
# Equivalence and Integrable Classes

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**Abstract**

A classification, according to invariant theory, of non-constant invariant Abel ODEs known as solvable and found in the literature is presented. A set of new integrable classes depending on one or no parameters, derived from the analysis of the works by Abel, Liouville and Appell [2–4], is also shown. Computer algebra routines were developed to solve any member of these classes by solving its related equivalence problem. The resulting library permits the systematic solving of Abel type ODEs in the Maple symbolic computing environment.

*Key words:*
First order ordinary differential equations of Abel type; Equivalence; Integrable classes; Symbolic computation.

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## PROGRAM SUMMARY

*Title of the software package: Extension to the Maple ODEtools package*

*Catalogue number:* (supplied by Elsevier)

*Software obtainable from:* CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue)

*Licensing provisions:* none

*Operating systems under which the program has been tested:* UNIX, Macintosh, Windows (95/NT).

*Programming language used:* Maple V Release 4 and 5

*Memory required to execute with typical data:* 16 Megabytes.

*Keywords:* Abel type first order ordinary differential equations (ODEs), equivalence problem, integrable cases, symbolic computation.

*Nature of mathematical problem*
Analytical solving of Abel type first order ODEs having *non-constant* invariant.

*Methods of solution*
Solving the equivalence problem between a given ODE and representatives of a set of non-constant invariant Abel ODE classes for which solutions are available.

*Restrictions concerning the complexity of the problem*
The computational scheme presented works when the input ODE belongs to one of the Abel classes considered in this work. This set of Abel classes can be extended, but there are classes - depending intrinsically on parameters - for which the equivalence problem as presented here may lead to large and therefore untractable expressions.

*Typical running time*
The methods being presented here have been implemented in the framework of the *ODEtools* Maple package. On the average, over Kamke's [1] first order Abel examples (see sec. 6), the ODE-solver of *ODEtools* is now spending $\approx$ 12 sec. per ODE when *successful*, and $\approx$ 22 sec. when *unsuccessful*. The timings in this paper were obtained using Maple R5 on a Pentium 200 - 128 Mb. of RAM - running Windows98.

*Unusual features of the program*
This computational scheme is able to integrate the infinitely many members of each of the Abel ODE classes presented here, all with *non-constant* invariant. When a given Abel ODE belongs to one of these solvable classes, the routines first determine this fact, without solving any differential equations, and use it to return a closed form solution without requiring further participation from the user. The ODE families that are covered include, as particular cases, all the Abel solvable cases presented in Kamke's and Murphy's books, as well as other Abel classes not previously presented in the literature to the best of our knowledge. After incorporating the new routines, the ODE solver of the ODEtools package succeeds in solving 96 % of Kamke's first order examples.

## LONG WRITE-UP

### 1 Introduction

From some point of view, after Riccati type ODEs, the simplest first order ordinary differential equations (ODEs) are those having as right hand side (RHS) a third degree polynomial in the dependent variable, also called *Abel type* ODEs [1]

$$y' = f_3\, y^3 + f_2\, y^2 + f_1\, y + f_0 \tag{1}$$

where $y \equiv y(x)$, and $f_0$, $f_1$, $f_2$ and $f_3$ are analytic functions of $x$. As opposed to Riccati ODEs, for which integration strategies can be built around their equivalence to second order linear ODEs, Abel ODEs admit just a few available integration strategies, most of them based on the pioneering works by Abel, Liouville and Appell around 100 years ago [2–4]. In those works it was shown that Abel ODEs can be organized into equivalence classes. Two Abel ODEs are defined to be equivalent if one can be obtained from the other through the general transformation which preserves the polynomial degree:

$$\{x = F(t), \qquad y(x) = P(t)\, u(t) + Q(t)\} \tag{2}$$

where $t$ and $u(t)$ are respectively the new independent and dependent variables, and $F$, $P$ and $Q$ are arbitrary functions of $t$ satisfying $F' \neq 0$ and $P \neq 0$.

Integration strategies were then discussed in [3,4], around objects *invariant* under Eq.(2) [2] (herein called the *invariants*) which can be built with the coefficients $\{f_3,\, f_2,\, f_1,\, f_0\}$ and their derivatives. To each class there corresponds a different set of values of these invariants, and actually any one of them (we shall pick one and call it the *invariant*) is enough to characterize a class. A simple integrable case happens when the invariant *is constant* [3]; the solution to the ODE then follows straightforwardly in terms of quadratures, as explained in textbooks [1,5]. On the contrary, when the invariant is not constant, just a few integrable cases are known and the formulation of solving strategies based on the equivalence between two such Abel ODEs, one of which is integrable, appears to be only partially explored in the literature, and not explored in general in computer algebra systems.

Having this in mind, this paper concerns Abel ODEs with non-constant invariant and presents:

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[1] For convenience, in this work, by "Abel ODEs" we mean Abel ODEs of first kind, since Abel ODEs of second kind can always be transformed into first kind by a simple change of variables.
[2] The invariants change in form for $F(t) \neq t$, but keep their value. See Eq.(5).
[3] There exists one invariant such that if it is constant then the other invariants are as well.

(1) A discussion and classification of the integrable Abel ODEs found both in Kamke's book and in the works from the late $19^{th}$ and early $20^{th}$ century by Abel, Appell, Liouville and other sources;
(2) A set of new integrable Abel ODE classes - some depending on arbitrary parameters - derived from those aforementioned works;
(3) An explicit method of verifying or denying the equivalence between two given Abel ODEs, one of which we know how to solve since it represents one of the above mentioned classes; and in the positive case, a way to determine the function parameters $F$, $P$ and $Q$ of the transformation Eq.(2) which maps one into the other;
(4) A computational alternative to the formulation of the equivalence problem in the case of *parameterized* classes; this includes the determination of the values for these parameters when a given Abel ODE is indeed a member of one of these classes;
(5) A set of computer algebra (Maple) routines implementing the algorithms presented in items (3) and (4) above, to systematically solve any Abel ODE belonging to one of the classes, parameterized or not, presented here and for which a closed form solution is known (items (1) and (2) above).

Item (1) is interesting since the Abel ODEs shown in Kamke's book and others are displayed without further classification, and in fact many of them belong to the same class. Actually we have found that the presentation of integrable Abel ODEs in textbooks in general is almost always done without a *classification* according to the classical invariant theory. The integrable classes mentioned in (2) are new to the best of our knowledge, although directly or indirectly derived from previous works. The formulation of the equivalence problem mentioned in (3) is the one given by Liouville in [3], is systematic and does not involve solving any auxiliary differential equations[4]. Concerning item (4), the idea can be viewed as a simple way of avoiding the untractable expressions which one would encounter by using Liouville's strategy directly for the case of *parameterized* solvable classes. The strategy presented is applicable only when there exists a solution for some numerical values of the parameter. Regarding item (5), the implementation presented here is, as far as we know, unique in computer algebra systems in its ability to solve complete non-constant invariant, parameterized or not, Abel ODE classes.

The paper is organized as follows. In sec. 2, the basic definitions and the classic formulation of the equivalence problem in terms of invariants is reviewed. In sec. 3 it is shown how these ideas can be complemented by taking advantage of computers to tackle the equivalence problem in the case of a parameterized class. Section 4 presents a classification of the integrable classes we have found in the literature with some additional comments as to their derivation. In sec. 5 new integrable Abel classes are presented. In sec. 6, a set of statistics is shown describing the performance in solving Kamke's first order examples using the ODE solver of the ODEtools package after incorporating the new routines. Finally, the conclusions contain some general remarks about this work and its possible extensions.

[4] An approach somewhat similar to this one by Liouville is discussed in [6].

Additionally, we present in the Appendix a table listing the distinct Abel ODE classes that we have found, representative ODEs from each class, and their respective solutions.

## 2 Classical Theory for Abel ODEs

In general, Abel type ODEs can be studied using two related concepts: *invariants* and ODE *equivalence classes*. We define two Abel ODEs to be equivalent [5] if one can be obtained from the other using a transformation of the form Eq.(2). The equivalence class containing a given ODE is then the set of all the ODEs equivalent to the given one. We note that although the infinitely many members of a class can be mapped into each other by using Eq.(2), there are also infinitely many disjoint Abel classes. Therefore Eq.(2) is not sufficient to map *any* Abel ODE into a given one.

To each class one can associate an infinite sequence of absolute invariants. To see this, consider two Abel ODEs, the first Eq.(1), the second obtained from Eq.(1) through the transformation Eq.(2)

$$u' = \tilde{f}_3\, u^3 + \tilde{f}_2\, u^2 + \tilde{f}_1\, u + \tilde{f}_0 \tag{3}$$

where the coefficients $\{\tilde{f}_0,\, \tilde{f}_1,\, \tilde{f}_2,\, \tilde{f}_3\}$, are related to the those of Eq.(1) by

$$\begin{aligned}
\tilde{f}_0 &= \frac{F' \left(f_2(F)\, Q^2 + f_1(F)\, Q + f_0(F) + f_3(F)\, Q^3\right) - Q'}{P} \\
\tilde{f}_1 &= \frac{P'}{P} - F' \left(2\, f_2(F)\, Q + 3\, f_3(F)\, Q^2 + f_1(F)\right) \\
\tilde{f}_2 &= F'\, P \left(f_2(F) + 3\, f_3(F) Q\right) \\
\tilde{f}_3 &= f_3(F) F'\, P^2
\end{aligned} \tag{4}$$

Following [4], we call an *absolute invariant* of Eq.(1) a function $I(f, x)$ of the coefficients $\{f_0,\, f_1,\, f_2,\, f_3\}$ and their derivatives with respect to $x$ such that, for all $\{F,\, P,\, Q\}$ in Eq.(2),

$$I(\tilde{f}, t)|_{\tilde{f}=\tilde{f}(f,t)} = I(f, x)|_{x=F(t)} \tag{5}$$

where $\tilde{f} = \tilde{f}(f, t)$ represents the coefficients $\{\tilde{f}_0,\, \tilde{f}_1,\, \tilde{f}_2,\, \tilde{f}_3\}$ and their derivatives with respect to $t$, expressed using Eq.(4).

[5] For a more formal definition of *class* see [7]

Similarly, we call a *relative invariant* a function, say $s$, of the coefficients of Eq.(1) and their derivatives such that when changing variables using Eq.(2), the resulting expression is equal to the original one up to a factor, say $\varphi_s$, dependent uniquely on the functions $F$, $P$, and $Q$ in Eq.(2) and independent of the coefficients themselves [8]:

$$s(\tilde{f})|_{\tilde{f}=\tilde{f}(f,t)} = \varphi_s(F,P,Q)\ s(f)|_{x=F(t)} \tag{6}$$

Liouville showed that in the case of Abel equations there is a relative invariant of weight 3

$$s_3 \equiv f_0 f_3^2 + \frac{1}{3}\left(\frac{2\,f_2^3}{9} - f_1 f_2 f_3 + f_3 f_2' - f_2 f_3'\right) \tag{7}$$

which can be used to recursively generate an infinite sequence of relative invariants $s_{2m+1}$ of weights $2m+1$ respectively [6], through the formula

$$s_{2m+1} \equiv f_3\, s'_{2\,m-1} - (2\,m-1)\, s_{2\,m-1}\left(f_3' + f_1\, f_3 - \frac{{f_2}^2}{3}\right) \tag{8}$$

As is clear from this definition, the product of two relative invariants respectively of weights $n$ and $m$ is a relative invariant of weight $n+m$, and by dividing any two relative invariants of equal weight one can generate an infinite sequence of absolute invariants

$$I_1 = \frac{s_5^3}{s_3^5},\ \ I_2 = \frac{s_7 s_3}{s_5^2},\ \ I_3 = \frac{s_9}{s_3^3},\ \text{etc...} \tag{9}$$

In [4], Appell showed that this sequence can also be obtained from two basic absolute invariants - say $J_1$, $J_0$, by expressing $J_1$ as a function of $J_0$ and then differentiating the result with respect to $J_0$. As a consequence, if $I_1$ is constant then all the other ones will be too. This fact allows one to identify the constant character of the invariants in Eq.(9) by looking at just the first one. We note also that there are infinitely many *different* classes having $I_1$ constant, related to the infinitely many possible constant values $I_1$ can have.

### *2.1 Integration strategy*

A description of a method of integration when the invariants are constant [7] is found in the works by Abel [2], Liouville [3] and Appell [4]. In such a constant invariant case, *all* members of the class can be mapped into a *separable* first order ODE by appropriately choosing $F$, $P$ and $Q$ in Eq.(2) (see for instance [1] and [5]).

[6] In the case of $s_3$, $\varphi_{s_3} = (F'P)^3$; the weight $n$ refers to the degree of $\varphi_{s_n}$ with respect to $(F'P)$.
[7] In [6] it is also shown that in the constant invariant case the problem can also be formulated in terms of the symmetries of these ODEs.

A quite different situation happens when $I_1$ is not constant. In such a case, relatively few integrable Abel ODEs are known, and the integration methods used to solve each of them depend in an essential way on non-invariant properties of the coefficients $f$. Those methods are then useless for solving the other infinitely many members of the same classes, unless one can solve the related equivalence problems; i.e., determining - when they exist -the values of $F$, $P$ and $Q$ in Eq.(2) linking two Abel ODE which belong to the same class.

### *2.2 Identifying an ODE as member of a given Abel ODE class*

Consider two Abel ODEs; the first one given by Eq.(1), and a second one being of the same form, but with coefficients $\tilde{f}_0$, $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$. The problem now is to determine whether the second Abel ODE can be obtained from Eq.(1) by changing variables using Eq.(2).

This problem can be formulated by equating the coefficients between the *transformed* equation, obtained by applying the transformation Eq.(2) to Eq.(1), and the second Abel ODE, resulting in Eq.(4), which can be seen as an ODE system for $\{F, P, Q\}$. To solve this system, following Liouville [3], we first note that the *absolute* invariants corresponding to the two Abel ODEs don't depend on $P$ or $Q$ (see previous section). Hence the function $F$ entering Eq.(2) can be obtained by just running an elimination process using two of these absolute invariants. Once $F$ is known, the system Eq.(4) becomes trivial in that $Q$ and $P$ can be re-expressed in terms of $F$ by performing fairly simple calculations. In the case of interest of this work - non-constant invariant [8] - the resulting expressions are:

$$P(t) = \frac{F' \, \tilde{f_3}^2 s_3}{{f_3}^2 \tilde{s_3}} \Big|_{x=F(t)} \qquad Q(t) = \frac{F' \tilde{f}_2 \, \tilde{f}_3 s_3 - f_2 f_3 \tilde{s_3}}{3 \, {f_3}^2 \tilde{s_3}} \Big|_{x=F(t)} \tag{10}$$

where $\{f_i, \tilde{f}_i\}$ with $i : 0 \rightarrow 3$ are the coefficients of the two Abel equations, $s_3$ is the relative invariant Eq.(7) expressed in terms of $f_i$ and $\tilde{s_3} = s_3|_{f_i = \tilde{f}_i}$. We now formulate the determination of $F$ as in [3], by calculating the absolute invariants $I_1 = s_5^3/s_3^5$ and $I_2 = s_5 s_7/s_3^4$, and setting up a system for $F$ with them:

$$0 = \frac{\tilde{s_5}^3}{\tilde{s_3}^5} - \frac{s_5^3}{s_3^5} \Big|_{x=F(t)} \qquad 0 = \frac{\tilde{s_5}\tilde{s_7}}{\tilde{s_3}^4} - \frac{s_5 s_7}{s_3^4} \Big|_{x=F(t)} \tag{11}$$

As discussed in [3,4], the *existence* of a common solution $F(t)$ to both equations above (such that $F' \neq 0$) is the necessary and sufficient condition for the existence of a transformation Eq.(2) relating the two Abel ODEs.

Concerning the explicit solution $F(t)$ for Eq.(11), we note that our interest in solving the equivalence problem is in that it leads directly to the solution of other members of an

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[8] When the invariant is non-constant, $s_3 \neq 0$.

Abel class, when the solution to a representative of the class is known. In turn, all the solvable classes we are aware of have a representative with rational coefficients, and hence also rational invariants [9]. Hence, assuming that one of the two Abel ODEs has rational coefficients and that Eq.(11) was obtained by applying Eq.(2) to it, the system Eq.(11) will always be polynomial in $F(t)$. In such a case, when a common solution $F(t)$ to both equations exists, the resultant between these polynomials will be zero [3]; i.e.: there will be a common factor, depending on $F$ and $t$ and representing the common solution, which can be obtained by calculating the greatest common divisor (GCD) between the two equations in Eq.(11). Conversely, if that GCD does not depend on $F$, a transformation Eq.(2) linking the input equation to Eq.(1) does not exist. That the dependence on $F$ of this GCD is a necessary condition for the existence of the desired transformation Eq.(2) is a consequence of the validity of Eq.(5) and hence the system (11). A proof of its sufficiency was given by Appell in [4].

The whole process just described to determine the equivalence between two given Abel ODEs, one of which is rational in $x$, can be summarized as follows:

(1) Calculate two absolute invariants, set up the system Eq.(11), and calculate the GCD between the two equations;
(2) When this GCD does not depend on $F$, the ODEs don't belong to the same class; otherwise determine an explicit expression for $F(t)$ from the result of the GCD calculation;
(3) Plug this value for $F$ into the formulas Eq.(10) to determine the values of $P(t)$ and $Q(t)$, arriving in this way at the transformation Eq.(2) mapping one Abel ODE into the other.

*Example:*

Consider the two non-constant invariant Abel ODEs

$$y' = -\frac{1}{2(x+4)}\left(x y^3 + y^2\right) \tag{12}$$

$$y' = \frac{(f'\,x - f)}{2\,(f + 3\,x)}\left((x - f)\,y^3 + y^2\right) - \frac{y}{x} \tag{13}$$

where in the above $f \equiv f(x)$ is an analytic (arbitrary) function. As in the typical situation one of these ODEs is rational in $x$ and we know its solution; i.e. for Eq.(12) we have

$$C_1 + \frac{\sqrt{y^2 x - 4\,y - 1}}{y} + 2\,\arctan\left(\frac{1 + 2\,y}{\sqrt{y^2 x - 4\,y - 1}}\right) = 0 \tag{14}$$

[9] On the other hand, there is no reason to expect that the second of the two Abel ODEs being tested for equivalence also has rational coefficients. If however *both* Abel ODEs are rational in $t$ and the coefficients are *numbers*, it is also possible to determine $F(t)$ by performing a rational function decomposition as mentioned in [6].

where $C_1$ is an arbitrary constant. We would then like to determine whether there are functions $\{F,\, P,\, Q\}$ so that Eq.(12) transforms under Eq.(2) into Eq.(13), and if so, determine $\{F,\, P,\, Q\}$ and use them together with Eq.(14) to build the answer to Eq.(13). For this purpose, we start (step (1)) by computing the relative invariants $s_3$, $s_5$, $s_7$, leading to Eq.(11)

$$\begin{aligned} 0 &= \frac{\left(87\, ft+9\, f^2+184\, t^2\right)^3}{t(9\, f+31\, t)^5} - \frac{\left(280+105\, F+9\, F^2\right)^3}{(40+9\, F)^5} \\ 0 &= \frac{\left(81\, f^3+1431\, f^2 t+7185\, ft^2+10903\, t^3\right)(9\, f+31\, t)}{\left(87\, ft+9\, f^2+184\, t^2\right)^2} - \frac{\left(1674\, F^2+10290\, F+81\, F^3+19600\right)(40+9\, F)}{\left(280+105\, F+9\, F^2\right)^2} \end{aligned} \tag{15}$$

where in the above $F \equiv F(t)$ is the function we are looking for and $f$ is taken at $x = t$. Calculating the GCD between the numerators of the expressions above (step (2)) and equating this GCD to zero, we obtain

$$27\,(t + tF - f) = 0 \tag{16}$$

from where the common solution $F(t)$ to both equations is given by

$$F(t) = \frac{f(t)}{t} - 1 \tag{17}$$

Substituting this value of $F$ into Eq.(10), a transformation of the form Eq.(2) mapping Eq.(12) into Eq.(13) is finally given by

$$\{x = \frac{f(t)}{t} - 1, \qquad y(x) = t\, u(t)\} \tag{18}$$

from where by changing variables in Eq.(14) using the transformation above and renaming the variables $(t \to x,\ u \to y)$, the solution to Eq.(13) is obtained

$$C_1 + \frac{\sqrt{\left(\frac{f}{x} - 1\right) x^2 y^2 - 4\, xy - 1}}{xy} + 2\, \arctan\left(\frac{(1 + 2\, xy)}{\sqrt{\left(\frac{f}{x} - 1\right) x^2 y^2 - 4\, xy - 1}}\right) = 0 \tag{19}$$

## 3 Parameterized Abel ODE classes

We formulate here this equivalence problem in the case of *parameterized* classes. By "parameterized class" we mean an (Abel) ODE class depending on symbolic parameters which *cannot be removed* by changing variables using Eq.(2). The interest in *parameterized* solvable classes is clear: to each set of values of the parameters corresponds - roughly speaking - a different Abel class [10]. Hence, a formulation for the equivalence problem of parameterized classes enables one to solve all the members of infinitely many classes at once.

In order to simplify the discussion, we first consider the problem of an Abel ODE class depending on just one parameter, say $\mathcal{C}$. Also, we distinguish between two different types of problems: one is when the equivalence problem has a solution for a specific *numerical* value of $\mathcal{C}$; the other happens when to have a solution it is required that $\mathcal{C}$ assumes *symbolic* values, for instance in terms of other *symbols* entering the input ODE. The discussion in this paper is restricted to the *numerical* case.

To facilitate the exposition we present the discussion around a concrete example. Consider the equivalence problem between a given Abel ODE, for instance,

$$y' = 8\,\frac{(1 - x^4 - x^8)\,y^3}{x^7} + 4\,\frac{y^2}{x^4} + \frac{y}{x} \tag{20}$$

and the one presented in Abel's memoires [2]

$$y' = \frac{(\mathcal{C}\,x^4 + x^2 + 1)\,y^3}{x^3} + y^2 \tag{21}$$

If this equivalence exists, then it exists just for one value of the parameter $\mathcal{C}$ since there is no solution for arbitrary $\mathcal{C}$ [11]. Hence, the common solution $F(t)$ to the system Eq.(11) will not show up until the correct value of $\mathcal{C}$ is determined, invalidating the itemized algorithm of the previous section.

A natural alternative to this problem would be to take one more absolute invariant, for instance, $s_3 s_7/s_5^2$, so that our system Eq.(11) becomes

$$0 = \frac{\tilde{s_5}^3}{\tilde{s_3}^5} - \frac{s_5^3}{s_3^5}\Big|_{x=F(t)} \qquad 0 = \frac{\tilde{s_3}\tilde{s_7}}{\tilde{s_5}^2} - \frac{s_3 s_7}{s_5^2}\Big|_{x=F(t)} \qquad 0 = \frac{\tilde{s_5}\tilde{s_7}}{\tilde{s_3}^4} - \frac{s_5 s_7}{s_3^4}\Big|_{x=F(t)} \tag{22}$$

Now, eliminate $\mathcal{C}$ from the first and second expressions above by taking the resultant with respect to $\mathcal{C}$, obtaining - say - $R_1$. In the same way, eliminate $\mathcal{C}$ from the first and the third expressions of Eq.(22) obtaining $R_2$. Hence, when a solution exists, the resultant between $R_1$ and $R_2$ with respect to $F$ will vanish. In other words, the algorithm of the previous section will work if instead of performing the calculations over the expressions Eq.(11) we perform them over $R_1$ and $R_2$, where $\mathcal{C}$ is already eliminated. The GCD between $R_1$ and $R_2$ will then return the factor depending on both $F$ and $t$, whose solution is the function $F(t)$ we are interested in. This method, simple and correct in theory, unfortunately does not work in

[10] There may be particular sets of values for which the resulting ODEs will belong to the same class.

[11] A solution for arbitrary $\mathcal{C}$ would mean the class by Abel does not really depend on any parameter.

practice because the expressions tend to grow in size so much that the computation of the first of these three resultants may not be possible, even with a simple example such as the one shown above. The problem resides in the fact that multivariate GCDs and resultants are quite expensive operations for the current symbolic computation environments.

The alternative we have found consists in reducing the problem to a sequence of bivariate GCD and resultant calculations, for which the available algorithms are relatively fast. The idea can be summarized as follows.

(1) From the previous considerations, when a solution to the equivalence problem exists, the resultant between any two of the expressions in Eq.(22) will not vanish for any value of $t$, since we haven't introduced the correct (unknown at this point) value of $\mathcal{C}$. Hence, if we insert in Eq.(22) a numerical value [12] for $t$ and calculate the GCD between any two of the resulting expressions, this GCD *cannot contain any factor depending on* $F$. This gives us a first "existence condition" test for the solution before proceeding further;

(2) When Eq.(22) evaluated at $t$ = number passed the test of the previous step, take two of the resulting three expressions and calculate their resultant with respect to $F$, obtaining, say, $\tilde{R}_1$. Then take a *different* pair and calculate their resultant with respect to $F$ again, obtaining, say, $\tilde{R}_2$. Neither of these resultants will vanish since the bivariate GCD calculations of the previous step showed no factor depending on $F$. Also, the calculation of $\tilde{R}_1$ and $\tilde{R}_2$ is now feasible since the expressions involve only the two unknowns $F$ and $C$;

(3) Then if a solution to the problem exists, the GCD between $\tilde{R}_1$ and $\tilde{R}_2$ will yield a factor depending on $\mathcal{C}$; equating it to zero and solving it for $\mathcal{C}$ will give the *common solution* $\mathcal{C}$ *for* $\tilde{R}_1$ *and* $\tilde{R}_2$. More precisely, what we will get in this way is a set of *candidates* (including among them the correct value) for $\mathcal{C}$; not all of them will necessarily lead to a solution $F(t)$ to the original problem;

(4) We now plug these candidates for $\mathcal{C}$ into Eq.(22), one at a time, receiving a system of three expressions involving again only two unknowns, now $F$ and $t$. If there is a common solution $F(t)$ to these expressions, the resultant with respect to F between any two of them will vanish. Hence, the GCD between those two expressions will contain a factor depending both on $F$ and $t$; equating this factor to zero and solving for $F$ leads to the solution $F(t)$.

Returning to our example of determining the equivalence between Eq.(20) and Eq.(21), the itemized procedure just outlined runs as follows.

According to step (1), $t = 0$ is tried first, but it is found to be an invalid evaluation point. The next value of $t$ to try, $t = 1$ turns out to be valid, so Eq.(22) was evaluated at $t = 1$; the GCDs between any two of the three resulting expressions do not depend on $F$, so this first test for the "existence" of a solution passed.

[12] We note there may exist "invalid evaluation points"; roughly speaking to avoid this problem this evaluation point must not cancel any of the coefficients of the variables remaining in the system - see [9].

Continuing with step (2), the calculation of $\tilde{R}_1$ and $\tilde{R}_2$ is performed without problems concerning the size of the expressions.

The GCD of step (3) results in the three factors: $36\,\mathcal{C}-5$, $\mathcal{C}+1$ and $9\,\mathcal{C}-2$; equating them to zero and solving them for $\mathcal{C}$ we arrive at three candidates for $\mathcal{C}$.

In step (4), plugging each of these candidates one at a time into Eq.(22) and taking the GCD between two of the three resulting expressions we note that $\mathcal{C} = 5/36$ does not lead to any factor depending on both $F$ and $t$, but $\mathcal{C} = -1$ leads to such factor: $F^2t^4 - 1$. So that for $\mathcal{C} = -1$ the problem admits two solutions: $F = \pm 1/t^2$

Finally, by introducing $F = 1/t^2$ into the formulas for $P$ and $Q$ Eq.(10) we arrive at the transformation of the form Eq.(2) mapping Eq.(21) into Eq.(20)

$$\left\{x = \frac{1}{t^2}, y = -2\,\frac{u(t)}{t}\right\} \tag{23}$$

and hence by applying the same change of variables to the answer of Eq.(21) *and* substituting $C = -1$ we obtain the answer to Eq.(20).

## 4 Integrable Abel ODE classes found in the literature

This section is devoted to a compilation of integrable Abel ODE classes found in the literature. The compilation is not intended to be complete, but it nevertheless covers various of the usual references; mainly Kamke's and Murphy's books [1,5], and the original works by Abel, Liouville and others on these subjects [2–4,11].

One of the noticeable things in these references is that the presentation of integrable cases lacks a classification in terms of their invariants. Consequently, many of these ODEs can actually be obtained from one another by means of Eq.(2), that is, they belong to the same class. Since part of this work consisted in writing computer routines addressing the equivalence problem, we performed this classification, and therefore present a more compact collection of integrable Abel ODE *classes*, as opposed to just integrable ODEs. Classes not depending on parameters are labelled by numbers (e.g., Class 1), while those depending on parameters are labelled with letters (e.g., Class A).

While revising the related literature we also noticed that various of the cases presented in books or papers are in fact particular cases of the integrable classes presented by Abel, Liouville and Appell in [2–4]. In turn the methods they used to obtain new integrable classes seem to be forgotten or not mentioned elsewhere. So, it appeared reasonable to start by reviewing and analyzing selected parts of those works in this section, and then show in the next section how, starting from these ideas, additional integrable classes can be obtained.

The first large presentation of integrable cases is due to Abel himself in [2]. His idea was to consider integrating factors of the form

$$\mu = \mathrm{e}^{r(x,y)} \tag{24}$$

for "Abel" equations written in terms of two arbitrary functions $p$ and $q$ as:

$$\Phi \equiv y y' + p(x) + q'(x) y = 0 \tag{25}$$

The first non-trivial case discussed in [2] was found by taking $r(x, y)$ as quadratic in $y$:

$$\mu = \mathrm{e}^{(\alpha+\beta\, y+\gamma\, y^2)}$$

where $\alpha$, $\beta$ and $\gamma$ are arbitrary functions of $x$. Abel formulated this problem by applying Euler's operator to the total derivative $\mu\Phi$, obtaining a system easily solvable for $\alpha$, $\beta$, $\gamma$ and $p$. The resulting Abel family has non-constant invariant and is shown in Abel's memoires as depending on one arbitrary function $q(x)$ and two arbitrary constants $C_i$:

$$y y' - \frac{q'}{2\, C_1\, q + C_2} + q'\, y \tag{26}$$

(for the corresponding integrating factor see [2]). Now, for the purpose of building computer routines addressing the equivalence problem, it is crucial to determine whether or not a given class depends on parameters since, as explained in sec. 3, in such a case the formulation of that problem is much more difficult. In the case of Eq.(26), the two parameters $C_i$ and the function $q(x)$, can be removed by first converting the ODE to first kind using $y(x) = 1/v(x)$, and then employing a transformation of the form Eq.(2): $\{x = F(t), v(x) = u(t)\sqrt{-2C_1}\}$, with $F$ implicitly defined by $2C_1 q(F) - t\sqrt{-2C_1} + C_2 = 0$, arriving at a representative of the class simpler than Eq.(26),

$$y' = \frac{y^3}{x} + y^2 \tag{27}$$

and showing that this class does not depend on parameters. It is then easy to verify that Eq.(27) is a particular case of a parameterized class [13] derived from Appell's work [4].

The next integrable case shown by Abel is obtained by considering for Eq.(25) an integrating factor of the form $\mu = \exp\left(1/(\alpha + \beta\, y)\right)$. Proceeding as in the previous case, Abel arrived at another integrable ODE class with non-constant invariant, which however (see [3]) is a particular member of the parameterized class Eq.(30) shown by Abel in the same paper.

*Constant Invariant case*

[13] Eq.(27) is obtained from Eq.(55) taking $C = 0$ and changing variables $\{x = i\,t, y = i\,u(t)\}$.

Abel then considered an integrating factor of the form $\mu = (\alpha + \beta\, y)^n$. This ansatz does not lead to a non-constant invariant family. However, this is the first presentation we have found of a method for the *constant* invariant case. Liouville, and others after him, rediscovered this method, presented in Kamke as due to M. Chini [10], and in Murphy's book as a change of variables mapping an Abel ODE into a separable one. A recent discussion of the symmetries of this *constant* invariant problem is found in [6].

*Class "A" depending on one arbitrary parameter*

The next ansatz considered by Abel was

$$\mu = (A+y)^a\,(B+y)^b\,y \tag{28}$$

where $A(x)$ and $B(x)$ are arbitrary functions and $a$ and $b$ are arbitrary constants. By taking $b=-a$ Abel showed that a tractable integrable case results:

$$yy' + \frac{q'}{4\,q}\left(\left(q+2\,\frac{C_1}{q}\right)^2 - \frac{q^2}{a^2}\right) + q'\,y = 0 \tag{29}$$

The arbitrary function $q(x)$ can be removed together with the constant $C_1$ by rewriting this ODE in first kind format, and then appropriately choosing $\{F,\,P,\,Q\}$ in Eq.(2); so that a simpler representative of this class depending on only one parameter "$\alpha$", is given by [14]

$$y' = \left(\alpha\,x + \frac{1}{x} + \frac{1}{x^3}\right)y^3 + y^2 \tag{30}$$

*Class 1*

In [11], Halphen noted a connection between doubly-periodic elliptic functions and the Abel type ODE

$$y' = \frac{3\,y\,(1+y) - 4\,x}{x\,(8\,y-1)} \tag{31}$$

which transforms into itself under infinitely many rational changes of variables, from where he was able to determine both a parametric and an algebraic solution for it. A simpler representative for this class and its solution can be found in the Appendix.

*Class 2*

In a paper by Liouville [3] mostly dedicated to Abel equations, he discussed the integrable cases known at that time (1903), and presented some new ones. Liouville reviewed Abel's

[14] A representative of the same class of Eq.(30) is shown in [3] as $y' = \frac{4}{9\,x^3}\left(\left(x^2+1\right)^2 - cx^4\right)y^3 + \frac{4\,y^2}{3}$

work and considered for Eq.(25) an integrating factor of the form Eq.(24) with $r(x,y)$ cubic in $y$, arriving at the integrable family $y' = 6\,a\,xy^2 + 3\,a\,y^3$, depending on a parameter $a$. This parameter however can be removed by changing variables as in $\{y = -u(t)/\sqrt[3]{3\,a}, x = t/\sqrt[3]{3\,a}\}$ arriving at the integrable class free of parameters represented by

$$y' = -2\,y^2 x + y^3 \tag{32}$$

<u>*Class "B" depending on one arbitrary parameter*</u>

As a generalization of Eq.(32), in [3] Liouville also presented the parameterized family

$$y' + \left(3\,mx^2 + 4\,m^2 x + n\right) y^3 + 3\,xy^2 = 0 \tag{33}$$

written in terms of two parameters $m$ and $n$ and which can be mapped into a Riccati ODE solvable in terms of special functions. Eq.(32) is a member of the class represented by Eq.(33) after setting $m = 0$. However, when $m = 0$, $n$ can be removed from Eq.(33) by changing variables $\left\{x = t\sqrt[3]{n},\ y(x) = t\,u(t)/n^{2/3}\right\}$, leading to a class without parameters - actually represented by Eq.(32). In turn, when $m \neq 0$, $m$ and $n$ can be "merged" by changing variables $\{y = u(t)/m^2, x = mt\}$ and introducing a new parameter $a = n/m^3$, resulting in

$$y' = -\left(3\,x^2 + 4\,x + a\right) y^3 - 3\,xy^2 \tag{34}$$

In summary, Eq.(33) is not a full 2-parameter class, but instead two classes represented by Eqs.(32) and (34), respectively depending on zero and one parameters. A simpler representative for this class and its solution can be found in the Appendix.

<u>*Class 3*</u>

Still in [3] Liouville pointed out that by interchanging the role between the dependent and independent variables in Eq.(32) one arrives at a different Abel integrable class. After rewriting this resulting ODE in first kind format and performing a change of variables of the form Eq.(2), a simpler representative of this integrable class is given by

$$y' = \frac{y^3}{4\,x^2} - y^2 \tag{35}$$

### *4.1 Integrable Abel ODE classes shown in Kamke and some others books*

One of the most well known collection of (69) Abel ODEs is the one shown in Kamke's book. This collection however makes no distinction between constant or non-constant invariant cases, presents ODEs of the same class as different, and does not discuss what would be the representative for each class depending on the least number of parameters. A first classification for these Abel ODEs is then given by[15]:

| Classification | ODE numbers as in Kamke's book |
|---|---|
| 4 are too general | 50, 219, 250, 269 |
| 43 constant invariant | 38, 41, 46, 49, 51, 51, 146, 169, 188, 204, 213, 214, 215, 216, 218, 221, 222, 223, 224, 225, 226, 227, 228, 229, 231, 236, 238, 239, 243, 244, 245, 246, 247, 248, 249, 251, 252, 254, 255, 260, 261, 262, 264 |
| 22 non-constant invariant | 36, 37, 40, 42, 43, 45, 47, 48, 111, 145, 147, 151, 185, 203, 205, 206, 234, 235, 237, 253, 257, 265 |
| 10 shown without solution | 40, 47, 48, 203, 205, 206, 234, 237, 253, 265 |

Table 1. First classification for the 69 Abel ODEs shown in Kamke's book.

As mentioned, all constant invariant ODEs can systematically be transformed into separable ODEs (see for instance Murphy's book), so that the interesting subset is the one comprising 22 ODEs having non-constant invariants. We note also that 10 of these 22 ODEs are shown in the book without a solution, and in fact we were unable to solve any of 203, 205, 206, 234, 253 or 265, so that the number of integrable cases for us is 16.

From these 16 ODEs (and hence from the 69 Abel type Kamke's examples), only four - those numbered: 47, 185, 235 and 237- would really lead to additional integrable classes with respect to those presented in the works by Abel, Liouville and Appell. We note however that the examples 47, 185 and 237 are all members of Class "C" (see Eq.(46)), which can be derived from the work by Abel [2] - even when it was not presented in the original work. So that the number of additional integrable classes presented in Kamke reduces to one, represented by the example 235. The classification and details are as follows.

*Class 4*

$$(xy + a)\, y' + b\, y = 0 \tag{36}$$

This ODE (K 1.235) is presented in Kamke in terms of two arbitrary parameters $\{a,\, b\}$; then, a change of variables which transforms it into a linear ODE is shown. A simpler representative of this class - not depending on parameters - can be obtained by rewriting this equation in first kind format via $\{x = t, y = \frac{1}{t\, u(t)} - \frac{a}{t}\}$ and then changing variables $\{x = \frac{a}{t\, b}, y = \frac{t\, u(t)}{a}\}$,

[15] In this classification, by "too general" we mean: these ODEs cannot be solved without restricting the example to a concrete particular case. We note also that the ODEs shown in Kamke without solution can all be transformed into an Emden type second order ODE presented in Kamke as 6.74, for which only a general discussion is presented. In turn, a detailed discussion on the integrable cases of Emden type ODEs is found in [12].

leading to

$$y' = y^3 - \frac{(x+1)}{x} y^2 \tag{37}$$

*Comments on Kamke's example 47*

For the ODE

$$y' - a\,(x^n - x)\,y^3 - y^2 = 0 \tag{38}$$

presented in Kamke as K 1.47, there is no solution shown in the book, but instead a suggestion of transforming the ODE into a second order one. We followed that suggestion and then ran a symmetry analysis, noticing that the resulting ODE will have two point symmetries if either $\{a = -\frac{2\,n+2}{9+6\,n+n^2}\}$ or $\{n = 2,\ a = \frac{6}{25}\}$, leading to two integrable classes not shown in the book. In the former case, from Eq.(38), we arrive at

$$y' + \frac{(2\,n+2)\,(x^n - x)\,y^3}{9 + n^2 + 6\,n} - y^2 = 0 \tag{39}$$

However, this ODE can be transformed into Eq.(46) by changing variables $\{x = t^{\frac{2}{1-n}}, y = -u(t)\frac{n+3}{2}t^{\frac{n+1}{n-1}}\}$ followed by $n = \frac{a+2}{a-2}$, so that it belongs to Class C. In the same line, taking $\{n = 2,\ a = \frac{6}{25}\}$ in Eq.(38), and changing variables $\{x = \frac{t^2-1}{t^2}, y = 5/2\,u(t)t^3\}$ one arrives at Eq.(46) with $a = 6$, so that this second branch of Eq.(38) is also a member of Class C.

*Comments on Kamke's example 237*

$$x\,(y + a)\,y' + b\,y + c\,x = 0 \tag{40}$$

This ODE (K 1.237) depending on three arbitrary parameters $\{a,\ b,\ c\}$, is presented in the book without a solution. We note however that changing $\{x \to y,\ y \to x\}$ leads to an ODE also of Abel type and in second kind format. Converting the latter to first kind format via $\{x = t, y = \frac{1}{c\,u(t)} - \frac{b\,t}{c}\}$, replacing $y \to y'$ and running a symmetry analysis, the resulting second order ODE has two symmetries when $a = -2\,b$, leading to an integrable case. Introducing $a = -2\,b$ into Eq.(40), rewriting it in first kind format via $\{x = t, y = -\frac{1}{t\,u(t)}+2\,b\}$ and changing variables $\{x = -\frac{b^2(t+4)}{2\,c}, y = \frac{2\,c\,u(t)}{b^3(t+4)}\}$ leads to a simpler representative of the class not depending on any parameters:

$$y' = \frac{-x\,y^3 + 2\,y^2}{2\,(x+4)} \tag{41}$$

However, by changing variables $\{x = 4\,(1-t^2)/t^2, y = -u(t)t/2\}$ one arrives at Eq.(46) again, this time with $a = -1/2$, so that Eq.(41) is also member of Class C.

A classification for all these 16 non-constant invariant Kamke examples is then as follows [16]

| Class 2 | Class 3 | Class 4 | Class A | Class B | Class C | Class D |
|---|---|---|---|---|---|---|
| 36, 40 | 145, 147 | 235 | 257 | 42, 43 | 45, 47, 48, 151, 185, 237 | 37, 111 |

Table 2. Classification for the 16 non-constant invariant solvable Abel ODEs in Kamke's book.

where classes C and D are defined in sec. 5. In summary, all but one of Kamke's 58 solvable examples (16 non-constant invariant + 42 constant invariant) are particular cases of the integrable classes presented by Abel, Liouville and Appell in [2–4], or can be derived from there (those belonging to Classes C and D).

Another collection of Abel ODEs is found in the book by Murphy [5]. After selecting those examples not having a constant invariant and for which a solution is shown in the book, we arrived at a set of nine ODEs, numbered in the book as: 78, 79, 80, 86, 275, 304, 345, 383 and 593. None of these ODEs represent an additional integrable class; their distribution among the classes discussed in this work is as follows

| Class 2 | Class 3 | Class B | Class C | Class D |
|---|---|---|---|---|
| 78, 80 | 275 | 86 | 304, 383, 593 | 79, 345 |

Table 3. Classification for the non-constant invariant solvable Abel ODEs in Murphy's book.

A wider collection of Abel ODEs than the one shown in Kamke's book is found in the book by Polyanin and Zaitsev [12]. This book is rather new (1995) and covers a vast number of integrable ODE problems which we have not found in other books, hence making the examples attractive. On the other hand the Abel ODEs shown there are classified not according to their invariants but according to their form, and the origin of their solutions is not given. Apart from a main section consisting of four tables (82 Abel ODEs - all derived from four basic ones), the book contains other sections illustrating mappings between Abel and higher order ODEs. The quantity of examples is large and the computational routines we prepared for the equivalence problem are not yet covering properly the case in which the parameters of the class may assume *symbolic* values. As a result we still don't have a way to solve the equivalence problem for the whole set of integrable classes presented in [12]. Our analysis of these Abel ODEs of [12] is then still incomplete; consequently we restricted the presentation here to just a sample, constituted by the ODEs of the first of these four tables. These are 20 ODEs obtained from

$$yy' - y = sx + Ax^m \tag{42}$$

by giving particular values to the parameters $m$ and $s$ ($A$ is kept arbitrary). These ODEs appear in section 1.3.1 of [12] under the numbers: 1, 2, 10, 16, 19, 22, 23, 26, 27, 30, 32, 33,

[16] Equations K.1.47, K.1.48 and K.1.237 belong to Class C for infinitely many - however particular - values of one of the two parameters (see Eq.(39)); we don't know their solution for other values.

45, 46, 47, 48, 53, 54, 55 and 56. We were not able to classify those numbered 27, 20, 48, 55 and 56. The distribution of the remaining ODEs, in the classes discussed in this work, is as follows:

| Constant invariant | Class 1 | Class 2 | Class 3 | Class C | Class D |
|---|---|---|---|---|---|
| 1, 2, 26 | 23 | 32 | 33 | 10, 19, 22, 45, 46, 47, 53, 54 | 16 |

Table 4. Classification for 15 of the 20 Abel ODE examples of Table 1.1 of [12].

## 5 New integrable Abel ODE classes derived from previous works

*Class "C" depending on one arbitrary parameter*

The form of the integrating factor studied by Abel actually leads to other integrable cases not mentioned in the original work [2]. One of them is obtained by taking $b = a$ in Eq.(28), resulting in the ODE family [17]

$$yy' - q'\,y - \frac{q'\,n^2\left(-\frac{q}{n} + {C_1}^2\left(\frac{q}{n}\right)^{2\,n-1}\right)}{(n+1)^2} = 0 \tag{43}$$

where $n \neq -1$. The function $q(x)$ and the parameter $C_1$ can be removed as done with Eq.(29), leading to

$$y' = n\left(x - x^{2\,n-1}\right) y^3 - (n+1)\,y^2 \tag{44}$$

which is turned exact by means of the integrating factor

$$\mu = \frac{\left(1 + \left(\left(x^2 - x^{2\,n}\right) y - 2\,x\right) y\right)^{-\frac{n+1}{2\,n}}}{y^{\frac{2\,n-1}{n}}} \tag{45}$$

A simpler representative of this class is obtained by changing variables $\{y = u(t)t^{\frac{n}{n-1}}, x = t^{\frac{1}{1-n}}\}$, then introducing a new parameter by means of $n = \frac{\alpha}{\alpha-2}$, arriving at

$$y' = \frac{\alpha\,(1 - x^2)\,y^3}{2\,x} + (\alpha - 1)\,y^2 - \frac{\alpha y}{2\,x} \tag{46}$$

[17] $n$ in Eq.(43) is related to $a$ in Eq.(28) by $n = -1/(2a + 1)$

Taking into account Eq.(45), an implicit solution for this class is given by

$$C_1 + \frac{\alpha}{x}\left(1 - \frac{(1-xy)^2}{y^2}\right)^{1/\alpha} - 2\int^{\frac{1-xy}{y}} \left(1 - z^2\right)^{\frac{1-\alpha}{\alpha}} dz = 0 \tag{47}$$

*Class "D" depending on one arbitrary parameter*

In [4], Appell showed a series of integrable cases derived from the solutions to

$$u' = A(u) + B(u)\, t \tag{48}$$

By changing variables $\{t = \frac{1}{y} - \frac{A(x)}{B(x)},\ u = x\}$, this ODE is transformed into the Abel ODE

$$y' = -\frac{y^3}{B} - \left(\frac{A}{B}\right)' y^2 \tag{49}$$

where $A$ and $B$ are now functions of $x$. Any particular $\{A,\, B\}$ leading to a solvable case in Eq.(48) will then also lead to an integrable Abel ODE Eq.(49). Among the choices for $\{A,\, B\}$ considered in [4] - such that Eq.(48) results linear, homogeneous, or of Riccati type - only this mapping into Riccati type leads to something new. This case is obtained by taking

$$A = ax^2 + bx + c, \qquad B = \alpha x^2 + \beta x + \gamma \tag{50}$$

The related Abel ODE family, depending on six parameters $\{a,\, b,\, c,\, \alpha,\, \beta,\, \gamma\}$, is given by

$$y' = -\frac{y^3}{\alpha\, x^2 + \beta\, x + \gamma} - y^2 \frac{d}{dx}\left(\frac{ax^2 + bx + c}{\alpha\, x^2 + \beta\, x + \gamma}\right) \tag{51}$$

and its solution could be expressed in terms of the solution to the Riccati ODE

$$y' = (a + \alpha x)\, y^2 + (b + \beta\, x)\, y + c + \gamma x \tag{52}$$

However, we were not able to solve this Riccati ODE for arbitrary values of the six parameters involved and in [4] there is no indication of how that could be done. The alternative we then investigated is to consider the second order ODE obtained by replacing $y = y'$ into Eq.(52). That ODE has *two* point symmetries if and only if $\alpha = 0$. With these symmetries we were able to solve that second order ODE, and hence Eq.(52) when $\alpha = 0$. Concerning the related Abel family Eq.(51) - now depending on five parameters - an appropriate change of variables of the form Eq.(2)

$$\left\{x = \frac{t\sqrt{\beta}}{a} - \frac{\gamma}{\beta},\quad y = \sqrt{\beta}\, u(t)\right\} \tag{53}$$

followed by the introduction of a new parameter $C$ by means of

$$C = -\frac{(\beta^2\, c + \alpha\, \gamma^2)\, a - \alpha\, \beta\, \gamma\, b}{\beta^2} \tag{54}$$

transforms Eq.(51) into a simpler representative for the class

$$y' = -\frac{y^3}{x} - \frac{(C + x^2)\, y^2}{x^2} \tag{55}$$

also showing that this class depends not on five but on one parameter [18] . It appeared of value to us also to determine the number of parameters on which Eq.(51) depends in the general case, that is *before* taking $\alpha = 0$. For that purpose we searched for the appropriate changes of variables of the form Eq.(2) which would remove as many as possible of these parameters, requiring that both the change and its inverse are finite. We then considered the branches which become infinite for some particular values of the parameters $\{a,\, b,\, c,\, \alpha,\, \beta,\, \gamma\}$ entering the transformations found. The results are summarized as follows. If all these parameters are different from zero, introducing new parameters $\{A,\, B,\, C,\, G\}$ by means of

$$\begin{aligned}
\alpha &= \frac{\beta^2 + 4\, A^4}{4\, \gamma}\\
b &= \frac{8\, \beta\, \gamma^2\, a\, A^2 B + C}{2\, \gamma\, A^2 B\, (\beta^2 + 4\, A^4)}\\
c &= \frac{A^2 B\, C + 16\, \gamma^2\, a\, A^6 B + \beta\, C + 4\, \beta^2 \gamma^2\, a\, A^2 B}{A^2 B\, (\beta^2 + 4\, A^4)^2}\\
\gamma &= \frac{C}{2\, A^3 B\, G\, (\beta^2 + 4\, A^4)}
\end{aligned} \tag{56}$$

followed by changing variables $\{x = \frac{C\left(2\, tA^2 - \beta\right)}{(\beta^2 + 4\, A^4)^2 A^3\, B\, G},\ y = u(t) A\}$ in the six-parameter Eq.(51), one arrives at a 2-parameter representative for the same class

$$y' = -\frac{y^3}{x^2 + 1} + \frac{G\, (B\, x + x^2 - 1)\, y^2}{(x^2 + 1)^2} \tag{57}$$

Now the case $\alpha = 0$ was already shown to lead to Eq.(55), and all the other possible branches

[18] We note that in this process we have made two implicit assumptions: $a \neq 0$ and $\beta \neq 0$. To assure that the cases in Eq.(51) are covered by Eq.(55) we then also considered $a = 0$ and $\beta = 0$ separately, arriving at ODEs respectively members of the classes represented by Eq.(55) and Eq.(32).

obtained from Eq.(51) lead either to constant invariant families, or to members of the classes already discussed in this work [19]

*Three new classes not depending on parameters*

While analyzing the works [2–4] and Kamke's examples, a large number of symbolic experiments were performed, sometimes leading to intermediate results which with a bit more of work appeared to be new integrable classes by themselves. This happened three times, resulting in classes 5, 6 and 7, for which representatives and solutions are given as follows:

*Class 5*

$$y' = -\frac{(2\,x+3)\,(x+1)\,y^3}{2\,x^5} + \frac{(5\,x+8)\,y^2}{2\,x^3} \tag{58}$$

Solution:

$$C_1 + \frac{\sqrt{A}}{\sqrt[4]{4\,\frac{(x+1)^2}{x^2\,A}+1}} + \int^{2\,\frac{x+1}{x\sqrt{A}}} \left(z^2+1\right)^{-5/4} dz = 0 \tag{59}$$

where $A = \frac{4}{y} - \frac{10}{x} - \frac{6}{x^2} - 4$.

*Class 6*

$$y' = -\frac{y^3}{x^2\,(x-1)^2} + \frac{(1-x-x^2)\,y^2}{x^2\,(x-1)^2} \tag{60}$$

Solution:

$$C_1 - \mathrm{Ei}\left(1, \frac{y+x^2-x}{xy\,(x-1)}\right) + \frac{(x-1)\;y\,\mathrm{e}^{\frac{x-y-x^2}{xy(x-1)}}}{x-1+y} = 0 \tag{61}$$

where $\mathrm{Ei}(n,x) = \displaystyle\int_1^\infty \frac{\mathrm{e}^{-xt}}{t^n} dt$ is the exponential integral.

*Class 7*

$$y' = \frac{(4\,x^2+1)\;y^3}{8\;(x^2+1)^{3/2}\,x^2} - \frac{\sqrt{x}\,y^2}{2\;(x^2+1)^{5/4}} \tag{62}$$

[19] There is a special case, when $b = 4\,\frac{\gamma\,\beta\,a}{\beta^2+4\,A^4}$, where the resulting Abel ODE can only be obtained from Eq.(57) by taking appropriate limits.

Solution:

$$C_1 + \frac{2\,x\,\sqrt[4]{x^2+1} + y\sqrt{x}}{x^{3/2}\,y\,\sqrt[4]{A^2+1}} + \int^{A} \left(z^2+1\right)^{-5/4} dz = 0 \tag{63}$$

where $A = \dfrac{2\,yx^2 + y + 2\,\sqrt[4]{x^2+1}}{x\left(2\,\sqrt[4]{x^2+1} - y\right)}$

## 6 Tests and performance

The two itemized algorithms described in sections 2.2 and 3 respectively for solving the equivalence problem between two given Abel ODEs were implemented in Maple R5, in the framework of the ODEtools package [13]. The implementation consists of various routines, mainly accomplishing the following:

(1) determine whether a given Abel ODE belongs to one of the solvable classes described in the previous sections; in doing that, determine also the function $F(t)$ entering Eq.(2) and the value of the parameters in the case of a parameterized class;
(2) use that information to determine the functions $P(t)$ and $Q(t)$ entering Eq.(2) and return a solution to the given ODE by means of changing variables in the solution available for the representative of the class.

The idea then was to test these computational routines to confirm the correctness of the returned solutions as well as to indirectly obtain the classification presented in the previous sections for solvable Abel ODEs. The testing arena was the 69 Abel examples found in Kamke plus the 9 solvable examples with non-constant invariant from Murphy's book, plus the 82 examples found in [12]. The routines passed these tests - the answers were confirmed to be correct using other symbolic computation tools - and the resulting classification is that shown in Tables 1, 2, 3 and 4 of sec. 4.

Also, a comparison of performances between the new routines and those available in other computer algebra systems (CAS) appeared to us not justified in this case: none of these CAS return solutions for Abel ODEs with non-constant invariant. As for the constant invariant case, only Maple R5 has implemented the corresponding routines to systematically transform these ODEs into separable, as explained for instance in Murphy's book.

### *6.1 Performance of the ODE-solver of ODEtools with the 1st order Kamke examples*

Although the main purpose of this paper is to present a computational scheme for finding solutions to Abel ODEs, it is interesting to see how **odsolve** - the ODE-solver of the

ODEtools Maple package [13] - performs with the addition of these new routines. The performance with all of Kamke's 555 solvable examples [20] after incorporating the computational routines presented in this paper is: 96 % are solved. This performance is summarized as follows

| Degree in $y'$ | ODEs | Solved | Average time | |
|---|---|---|---|---|
| | | | *solved* | *fail* |
| 1 | 350 | 337 | 3.2 sec. | 12.9 sec. |
| 2 | 147 | 140 | 8.8 sec. | 61.1 sec. |
| 3 | 27 | 26 | 7.2 sec. | 17 sec. |
| higher | 31 | 30 | 13.4 sec. | 25.2 sec. |
| Total: | 555 | 533 | $\approx$ 6 sec. | $\approx$ 20 sec. |

Table 5. *Kamke's first order ODEs, solved by* **odsolve***:* 96%

The number and classification of Kamke's 1$^{\text{st}}$ order ODEs still not solved by **odsolve** is now:

| Class | Kamke's numbering |
|---|---|
| rational | 452, 480, 485 |
| Riccati | 25 |
| Abel | 234, 253, |
| NONE | 80, 81, 83, 87, 121, 128, 340, 367, 395, 460, 506, 510, 543, 572 |

Table 6. *Kamke's 1$^{\text{st}}$ order solvable ODEs for which* **odsolve** *fails:* 4%

where the Abel ODEs numbered in Kamke's book as 47, 48, 205, 206, 237, 253 and 265 not presented in the tables above are known to be solvable only for specific values of their parameters - not in general. Also, for the Abel ODEs 234 and 253 not depending on parameters and included in the table of failures above, the solution is not shown in the book or known to us.

## 7 Conclusions

In this paper, a first classification, according to invariant theory, of solvable non-constant invariant Abel ODEs found in the literature, was presented. Also, a set of new solvable classes, depending on one or no parameters, derived from the analysis of the works by Abel,

[20] We classified as *unsolvable* in general Kamke's examples 50, 55, 56, 74, 79, 82, 202, 219, 250, 269, 331, 370, 461, 503 and 576.

Liouville and Appell [2–4], was shown. Computer algebra routines were then developed, in the framework of the Maple ODEtools package, to solve any member of these classes by solving its related equivalence problem. The result is a whole new tool that extends, in a symbolic computing environment, our capacity to solve Abel type ODEs.

The classification shown has had the intention of giving a first step towards organizing in a single reference the integrable cases widely scattered throughout the literature. The derivation of new solvable parameterized classes from the works by Abel and Appell in the $19^{th}$ century (Classes "C" and "D") also showed that valuable information can still be obtained from these old papers. In fact, from Tables 1, 2, 3 and 4 in sec. 4, the larger number of integrable cases found in textbooks are particular members of this Class "C" (Eq.(46)) - an integrable class derived by considering a case somehow disregarded in Abel's Memoires [2].

As for the computer routines, the implementation presented here for solving the equivalence problem for *parameterized* classes - when the parameters assume *numerical* values - proved to be a valuable tool in most of the Abel ODE examples we were able to collect. In fact these routines were crucial in detecting the large number of cases presented as different in the literature, but actually being members of the same class.

Another thing worth mentioning concerning these computer routines is that almost none of the computer algebra systems available have implemented methods for this relevant problem of Abel ODEs. As far as we know, only Maple R5 has specialized routines working in the framework of invariant theory, but just for the easy case in which the Abel ODEs have constant invariants - the method being that presented in the books by Murphy and Kamke. In turn, it is our belief that computer algebra systems can bring relevant advantages for tackling these types of problems when the invariants are not constant, as shown in this work.

On the other hand, we note the intrinsic limitation of this Abel problem: most of the answers can only be obtained in implicit form and in terms of quadratures; in turn, these integrals are usually elliptic integrals so that they cannot be expressed using elementary functions. Also the collection presented here is incomplete in that it is missing - at least - a more thorough analysis of the integrable cases presented in [12].

Finally, concerning the more difficult task of solving the equivalence problem when the parameters assume *symbolic* values, related routines are presently under development [14]; we expect to succeed in obtaining reportable results in the near future.

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## Appendix A[22]

| Class | Representative Equation and Solution |
|---|---|
| 1 | $y' = \frac{3\,y^2-3\,y-x}{x(8\,y-9)},\ C_1 + \frac{x^3\left(4\,x^2+\left(8\,y^2-36\,y+27\right)x+4\,y^4-4\,y^3\right)}{(x^2+2\,x(y^2-3\,y)+y^4)^3} = 0$ |
| 2 | $y' = -2\,y^2x + y^3,\ C_1 + \frac{x\,\mathrm{Ai}\left(x^2-\frac{1}{y}\right)+\mathrm{Ai}\left(1,x^2-\frac{1}{y}\right)}{x\,\mathrm{Bi}\left(x^2-\frac{1}{y}\right)+\mathrm{Bi}\left(1,x^2-\frac{1}{y}\right)} = 0$ |
| 3 | $y' = \frac{y^3}{4\,x^2} - y^2,\ C_1 + \frac{\left(x-\frac{1}{y}\right)\mathrm{Ai}\left(\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)+\mathrm{Ai}\left(1,\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)}{\left(x-\frac{1}{y}\right)\mathrm{Bi}\left(\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)+\mathrm{Bi}\left(1,\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)} = 0$ |
| 4 | $y' = y^3 - \frac{x+1}{x}y^2,\ C_1 + \frac{1}{x}e^{\frac{1}{y}-x} - Ei(1, x - \frac{1}{y}) = 0$ |
| 5 | $y' = -\frac{(2\,x+3)(x+1)y^3}{2\,x^5} + \frac{(5\,x+8)y^2}{2\,x^3}$ <br> $C_1 + \frac{\sqrt{A}}{\sqrt[4]{4\,\frac{(x+1)^2}{x^2\,A}+1}} + \int^{2\,\frac{x+1}{x\sqrt{A}}} \left(z^2+1\right)^{-5/4} dz = 0, A = \frac{4}{y} - \frac{10}{x} - \frac{6}{x^2} - 4$ |
| 6 | $y' = -\frac{y^3}{x^2(x-1)^2} + \frac{\left(1-x-x^2\right)y^2}{x^2(x-1)^2}\ C_1 - \mathrm{Ei}\left(1, \frac{y+x^2-x}{xy(x-1)}\right) + \frac{(x-1)\,y\,\mathrm{e}^{\frac{x-y-x^2}{x\,y\,(x-1)}}}{x-1+y} = 0$ |
| 7 | $y' = \frac{\left(4\,x^2+1\right)y^3}{8\,(x^2+1)^{3/2}x^2} - \frac{\sqrt{x}\,y^2}{2\,(x^2+1)^{5/4}}$ <br> $C_1 + \frac{2\,x\,\sqrt[4]{x^2+1}+y\sqrt{x}}{x^{3/2}\,y\,\sqrt[4]{A^2+1}} + \int^{A} \left(z^2+1\right)^{-5/4} dz = 0, A = \frac{2\,yx^2 + y + 2\,\sqrt[4]{x^2+1}}{x\left(2\,\sqrt[4]{x^2+1} - y\right)}$ |
| A | $y' = \left(\alpha\,x + \frac{1}{x} + \frac{1}{x^3}\right)y^3 + y^2,$ <br> $C_1 + \frac{x^3}{y+x}\,exp\left(\int^{\frac{-yx^2}{y+x}} \frac{2\,dz}{z^2-z-\alpha z^3}\right) - \int^{-\frac{-yx^2}{y+x}} exp\left(\int \frac{2\,dz}{z^2-z-\alpha z^3}\right) dz = 0$ |
| B | $y' = 2\,\left(x^2-\alpha^2\right)y^3 + 2\,\left(x+1\right)y^2,$ <br> $C_1 + \frac{-(\alpha+x)\mathrm{K}\left(\alpha,-\sqrt{-\alpha^2+x^2+\frac{1}{y}}\right)-\sqrt{-\alpha^2+x^2+\frac{1}{y}}\ \mathrm{K}\left(1+\alpha,-\sqrt{-\alpha^2+x^2+\frac{1}{y}}\right)}{-(\alpha+x)\mathrm{I}\left(\alpha,-\sqrt{-\alpha^2+x^2+\frac{1}{y}}\right)+\sqrt{-\alpha^2+x^2+\frac{1}{y}}\ \mathrm{I}\left(1+\alpha,-\sqrt{-\alpha^2+x^2+\frac{1}{y}}\right)} = 0$ |
| C | $y' = \frac{\alpha\left(1-x^2\right)y^3}{2\,x} + (\alpha-1)\,y^2 - \frac{\alpha y}{2\,x},\ C_1 + \frac{\alpha}{x}\left(1 - \frac{(1-xy)^2}{y^2}\right)^{1/\alpha} - 2\int^{\frac{1-xy}{y}} \left(1-z^2\right)^{\frac{1-\alpha}{\alpha}} dz = 0$ |
| D | $y' = -\frac{y^3}{x} - \frac{\left(\alpha+x^2\right)y^2}{x^2},$ <br> $C_1 + \frac{(\alpha+1)\mathrm{M}\left(-\frac{\alpha}{2}-\frac{3}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)+\left(\frac{x}{y}-x^2\right)\mathrm{M}\left(-\frac{\alpha}{2}+\frac{1}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)}{(\alpha^2+\alpha)\mathrm{W}\left(-\frac{\alpha}{2}-\frac{3}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)+2\left(\frac{x}{y}-x^2\right)\mathrm{W}\left(-\frac{\alpha}{2}+\frac{1}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)} = 0$ |

Representative ODEs and their solutions for the Abel ODE classes presented in this work.

[22] The solution shown for the representative of class D is not valid when $\alpha$ is an integer, or when $2\alpha$ is a positive integer. In those cases, the solution of the associated Riccati equation Eq.(52) takes many different forms depending on the value of $\alpha$, which we found inconvenient to present here. $\mathrm{Ei}(n,x) = \displaystyle\int_1^\infty \frac{\mathrm{e}^{-xt}}{t^n} dt$ is the exponential integral, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy wave functions, $\mathrm{K}(x)$ and $\mathrm{I}(x)$ are the modified Bessel functions of the first and second kinds, respectively, and $\mathrm{M}(x)$ and $\mathrm{W}(x)$ are the Whittaker functions.