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An Abel ODE class generalizing known integrable classes

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A multi-parameter non-constant invariant Abel ODE class with the following remarkable features is presented. This one class is shown to generalize, that is, contain as particular cases, all the integrable classes presented by Abel, Liouville and Appell, as well as all those shown in Kamke's book and various other references. In addition, the class being presented includes other new and fully integrable subclasses, as well as the most general parameterized class we know of whose members can systematically be mapped into Riccati type ODEs. Finally, many integrable members of this class can be systematically mapped into an integrable member of a *different* class; leading in this way to new integrable classes from previously known ones.

1 Introduction

Abel type ODEs of first kind are polynomial first order ODEs of the form

$$y' = f_3\, y^3 + f_2\, y^2 + f_1\, y + f_0 \tag{1.1}$$

where $y \equiv y(x)$ and f_i ($i :\rightarrow 1$ to 3) are in principle arbitrary functions of x. Abel equations appear in the reduction of order process related to finding exact solutions to many second and higher order ODE families [6, 11, 12], and hence are frequently found in the modelling of real problems in varied areas. That has for a long time motivated people to study their integrable cases and related solving methods. A general "exact integration" strategy for these ODEs was first formulated by Liouville [7] and is based on the concepts of classes, invariants and the solving of the equivalence problem [2, 3, 8]. Generally speaking, two Abel ODEs of first kind belong to the same equivalence class if and only if one can be obtained from the other by means of a transformation of the form

$$\{x = F(t), \quad y = P(t)\, u + Q(t)\} \tag{1.2}$$

where t and $u \equiv u(t)$ are respectively the new independent and dependent variables, and F, P and Q are arbitrary functions satisfying $F' \neq 0$ and $P \neq 0$. By changing variables $\{x = t,\ y = (g_1 u + g_0)^{-1}\}$, where $\{g_1, g_0\}$ are arbitrary functions of t, Abel ODEs of first kind can always be written in second kind format

$$y' = \frac{\tilde{f}_3\, y^3 + \tilde{f}_2\, y^2 + \tilde{f}_1\, y + \tilde{f}_0}{g_1\, y + g_0} \tag{1.3}$$

Abel ODEs of second kind belong to the same class as their first kind partners. However, due to the arbitrariness introduced when switching from first to second kind form, the transformation preserving the class for the latter form becomes

$$\{x = F(t), \quad y = \frac{P_1(t)\, u + Q_1(t)}{P_2(t)\, u + Q_2(t)}\} \tag{1.4}$$

where $P_1 Q_2 - P_2 Q_1 \neq 0$. There are infinitely many Abel ODE classes, and their classification is performed by means of algebraic expressions invariant under Eq.(1.2) - the so called invariants - built with the coefficients f_i in Eq.(1.1) and their derivatives.

When the invariants of a given Abel ODE are constant its integration is straightforward: the ODE can be transformed into a *separable* ODE as explained in textbooks [9]. On the contrary, when the invariants are non-constant, the integration strategy relies on recognizing the ODE as equivalent to one of a set of previously known integrable equations, and then applying the equivalence transformation over that known solution. However, for non-constant invariant Abel ODEs, only a few integrable classes are known. In a recent work [3], for instance, a classification of all integrable cases presented by Abel, Liouville and others [1, 2, 5, 8], including examples from Kamke's book, showed - in all - only four classes depending on one parameter and seven classes depending on no parameters.

In this work, a single *multi-parameter* Abel ODE class (AIA[1]) generalizing all the integrable cases collected in [3] is presented. In addition, AIA contains as particular case a new subclass (AIR), depending on 6 parameters, all of whose members can be systematically transformed into Riccati type ODEs. This AIR class, in turn, includes a 4-parameter fully integrable subclass (AIL). Finally, some of the members of this AIA class (including all those not in the AIR subclass) can be mapped into non-trivial Abel ODEs belonging to a *different* class. Hence, if the member being mapped is solvable, it can be used to generate a different - maybe new - solvable and non-constant invariant class (see sec. 3).

Due to its simplicity and the potential preparation of computer algebra routines for solving the related equivalence problem [3, 4], the material being presented appears to us as a convenient starting point for a more systematic determination of exact solutions for Abel equations.

2 The AIL, AIR and AIA classes

As mentioned in the introduction, the whole collection of integrable classes presented in [3], consisting of four 1-parameter classes and 7 classes without parameters, can be obtained by assigning particular values to the parameters of a single multi-parameter Abel class, AIA. In turn, AIA contains a subclass - AIR - all of whose members can

[1] The acronyms AIA, AIR and AIL are explained below.

be transformed into Riccati type equations, and inside AIR there is a fully integrable subclass - AIL - all of whose members can be mapped into first order linear ODEs.

Since all these classes are obtained by means of the same procedure, to better illustrate the ideas we discuss first this AIL (Abel, Inverse-Linear) subclass. So consider the general form of a first order linear ODE $y' + g(x)\, y + f(x) = 0$, where f and g are in principle arbitrary, after changing variables by means of the *inverse* transformation $\{x \leftrightarrow y\}$[2]:

$$y' = -\frac{1}{g(y)x + f(y)}. \tag{2.1}$$

An implicit solution to this equation is easily expressed in terms of quadratures as

$$C_1 = x \, \exp\left(\int g(y)\ dy\right) + \int \exp\left(\int g(y)\ dy\right) f(y)\ dy \tag{2.2}$$

The key observation here is that Eq.(2.1) will be of type Abel second kind for many choices of f and g. For instance, by taking

$$f(y) = \frac{s_0 y + r_0}{a_3\, y^3 + a_2\, y^2 + a_1\, y + a_0}, \qquad g(y) = \frac{s_1 y + r_1}{a_3\, y^3 + a_2\, y^2 + a_1\, y + a_0}, \tag{2.3}$$

where $\{s_1,\, s_0,\, r_1,\, r_0,\, a_i\}$ are arbitrary constants, the resulting Abel family is

$$y' = -\frac{a_3\, y^3 + a_2\, y^2 + a_1\, y + a_0}{(s_1\, x + s_0)\; y + r_1\, x + r_0} \tag{2.4}$$

This ODE has non-constant invariant, can then be seen as a representative of a non-trivial multi-parameter class, and from its connection with a linear ODE, its general solution is obtained directly from Eqs.(2.2) and (2.3). As shown in the next section, four of the eight parameters in Eq.(2.4) are superfluous. Even so, the class is surprisingly large, including multi-parameter subclasses we have not found elsewhere[3]. Among others, the class represented by Eq.(2.4) contains as particular cases the two integrable 1-parameter classes related to Abel's work [1, 3], and most of the examples found in Kamke's as well as in other textbooks [11].

We note that if instead of starting with a linear ODE we were to start with a Bernoulli ODE, instead of Eq.(2.4) we would obtain

$$y' = -\frac{a_3 y^3 + a_2 y^2 + a_1 y + a_0}{(s_1 x + s_0 x^\lambda)\, y + r_1 x + r_0 x^\lambda} \tag{2.5}$$

which is reducible to Eq.(2.4) by changing $\{x = t^{1/(1-\lambda)},\ y = u\}$ followed by redefining the constants $c_i \rightarrow (1-\lambda)\, a_i$, and so it belongs to the same class as Eq.(2.4). However, if

[2] By $\{x \leftrightarrow y\}$ we mean changing variables $\{x = u,\ y = t\}$ followed by renaming $\{u \rightarrow y,\ t \rightarrow x\}$.

[3] We noted afterwards, however, that Eq.(2.4) could also be obtained using a different approach, for instance by following Olver [10] and considering Eq.(1.1) as an "inappropriate reduction" of a second-order ODE that *has* a solvable non-Abelian Lie algebra; the resulting restrictions on the coefficients in Eq.(1.1) surprisingly lead to an ODE family of the same class as Eq.(2.4).

instead of starting with a Bernoulli ODE we start with a Riccati ODE, and hence instead of Eq.(2.1) we consider

$$y' = -\frac{1}{h(y)\,x^2 + g(y)\,x + f(y)} \tag{2.6}$$

and then choose $f(y)$, $g(y)$ and $h(y)$ as in Eq.(2.3), we obtain a 10-parameter $\{s_i,\, r_i, a_k\}$ Abel type family (AIR, meaning Abel, Inverse-Riccati)

$$y' = -\frac{a_3 y^3 + a_2 y^2 + a_1 y + a_0}{(s_2 x^2 + s_1 x + s_0)\,y + r_2 x^2 + r_1 x + r_0} \tag{2.7}$$

which becomes a Riccati type ODE by changing $\{x \leftrightarrow y\}$. Four of these 10 parameters are superfluous. The remaining 6-parameter class includes as particular cases the parameterized families presented by Liouville and Appell, as well as other classes depending on no parameters shown in Kamke and [11] and having solutions in terms of special functions. We note that Eq.(2.4) is also a particular case of Eq.(2.7).

Finally, the same ideas can be used to construct a more general Abel class (AIA) embracing the previous classes Eqs.(2.4) and (2.7) as particular cases. This AIA family is obtained by taking as starting point an Abel ODE of second kind, so that instead of Eq.(2.7) one obtains

$$y' = -\frac{(b_3 x + a_3)\,y^3 + (b_2 x + a_2)\,y^2 + (b_1 x + a_1)\,y + b_0 x + a_0}{(s_3 x^3 + s_2 x^2 + s_1 x + s_0)\,y + r_3 x^3 + r_2 x^2 + r_1 x + r_0} \tag{2.8}$$

Remarkably, by changing $\{x \leftrightarrow y\}$ one arrives at an ODE of exactly the same type

$$y' = -\frac{(s_3 x + r_3)\,y^3 + (s_2 x + r_2)\,y^2 + (s_1 x + r_1)\,y + s_0 x + r_0}{(x^3 a_3 + x^2 a_2 + a_1 x + a_0)\,y + x^3 b_3 + x^2 b_2 + x b_1 + b_0} \tag{2.9}$$

Therefore, when applied to representatives of solvable subclasses of AIA, the change of variables $\{x \leftrightarrow y\}$ may lead to representatives of *new* non-trivial Abel solvable classes (see sec. 2.3). Due to this feature and since AIA already contains as a particular case the AIR class and hence AIL (Eqs.(2.7) and (2.4)), this AIA class generalizes in one all the solvable classes presented in [3], collected from the literature (see sec. 3).

2.1 The intrinsic parameter dependence of the classes AIL, AIR and AIA

For the purposes of finding exact solutions to Abel ODEs, and in doing that solving the equivalence problem with respect to the classes represented by Eqs.(2.4) and to some extent (2.7), it is relevant to determine, precisely, on *how many parameters* these ODE classes intrinsically depend. In fact, the technique we have been using for tackling the equivalence problem relies heavily on the calculation of multivariate resultants, and even for classes depending on just one parameter these calculations require the use of special techniques to become feasible [4].

The number of parameters in the AIL and AIR classes can be reduced by four by performing almost the same steps, exploring fractional linear transformations. We illustrate here with the simplest case, AIL (see Eq.(2.4)) - originally depending on 8 parameters. So by changing variables[4] in Eq.(2.4) according to

$$\left\{ x = -\frac{t}{{s_1}^2},\; y = -\frac{r_0 {s_1}^2 + r_1 u}{(s_1 s_0 + u)\, s_1} \right\} \tag{2.10}$$

and introducing new parameters $\{k_0,\, k_1,\, k_2,\, k_3\}$ according to

$$k_0 = \frac{\left(\left(a_2\, {r_0}^2 + (s_0 a_0 - a_1\, r_0)\, s_0\right) s_0 - a_3\, {r_0}^3\right) {s_1}^2}{\left(r_1 s_0 - r_0 s_1\right)^2} \tag{2.11}$$

$$k_1 = \frac{(a_2\, s_1 - 3\, a_3\, r_1)\, {r_0}^2 + \left((2\, r_1 a_2 - 2\, a_1\, s_1)\, r_0 + (3\, a_0\, s_1 - a_1\, r_1)\, s_0\right) s_0}{\left(r_1 s_0 - r_0 s_1\right)^2} \tag{2.12}$$

$$k_2 = \frac{\left((2\, a_2\, s_1 - 3\, a_3\, r_1)\, r_1 - a_1\, {s_1}^2\right) r_0 + \left(3\, {s_1}^2 a_0 + (r_1 a_2 - 2\, a_1\, s_1)\, r_1\right) s_0}{{s_1}^2 \left(r_1 s_0 - r_0 s_1\right)^2} \tag{2.13}$$

$$k_3 = \frac{a_0\, {s_1}^3 + \left((a_2\, s_1 - a_3\, r_1)\, r_1 - a_1\, {s_1}^2\right) r_1}{{s_1}^4 \left({r_0}^2 {s_1}^2 + ({r_1}^2 s_0 - 2\, r_1 r_0 s_1)\, s_0\right)} \tag{2.14}$$

Eq.(2.4) is transformed into

$$y' = \frac{k_3 y^3 + k_2 y^2 + k_1 y + k_0}{y + x} \tag{2.15}$$

whose solution can be obtained directly from Eqs. (2.2) and (2.3) by taking $s_1 = 0$, $r_1 = 1$, $s_0 = 1$, $r_0 = 0$ and $a_i \to -k_i$. For classification purposes (see sec. 3), it is convenient to write this ODE in first kind form by changing variables $\{x = t, y = \frac{1}{u} - t\}$, leading to

$$y' = \left(k_3 x^3 - k_2 x^2 + k_1 x - k_0\right) y^3 - \left(3 k_3 x^2 - 2 k_2 x + k_1 + 1\right) y^2 + \left(3 k_3 x - k_2\right) y - k_3 \tag{2.16}$$

A reduction of the number of parameters, equivalent to this one performed over AIL can be performed over Eq.(2.7) leading to a representative of the same class depending on only 6 parameters:

$$y' = \frac{k_3\, y^3 + k_2\, y^2 + y k_1 + k_0}{(\tilde{s}_2 x^2 + 1)\, y + \tilde{r}_2 x^2 + x} \tag{2.17}$$

Regarding Eq.(2.8), we noted that any possible removal of parameters seems to lead to a “splitting into cases”, related to singularities in the changes of variables used. In turn, a full splitting of Eq.(2.8) into cases does not appear to us of concrete interest for the purpose of this work.

[4] We are assuming $r_1 s_0 - r_0 s_1 \neq 0$ and $s_1 \neq 0$, which is justified, since in the first case Eq.(2.4) has *constant invariant*, therefore presenting no interest, and in the second case it can be transformed into an ODE of the form Eq.(2.15) anyway by means of $\{x = t/r_1, y = (u - r_0)/s_0\}$.

2.2 Splitting of AIL into cases

Since Eq.(2.15) is *fully* solvable, it makes sense, for the purpose of tackling its related equivalence problem, to consider the maximum further reduction in the number of parameters, and hence completely split the class into a set of non-intersecting subclasses, all of which depend intrinsically on a minimum number of parameters. With that motivation we performed some algebraic manipulations, finally determining that Eq.(2.15) actually can be seen to consist of two different classes, respectively depending on two and one parameters.

Case $k_3 \neq 0$

In this case, by redefining $k_3 \equiv -{k_4}^2$, then changing variables in Eq.(2.15) according to

$$\{x = -\frac{k_2 + 3\,t\,k_4}{3\,{k_4}^2}, y = -\frac{1}{k_4\,u} + \frac{k_2 + 3\,t\,k_4}{3\,{k_4}^2}\} \tag{2.18}$$

and next redefining $\{k_4, k_0, k_1\}$ in terms of new parameters $\{\alpha,\, \beta,\, \gamma\}$ according to

$$k_4 = -\frac{\beta}{\gamma}, \qquad k_0 = \frac{{k_2}^3\gamma^4}{27\beta^4} - \frac{k_2\alpha\gamma^2}{3\beta^2} + \gamma, \qquad k_1 = \alpha - \frac{{k_2}^2\gamma^2}{3\beta^2} \tag{2.19}$$

we obtain a 2-parameter representative of the class, already in first kind format:

$$y' = \left(x\alpha - \beta - x^3\right) y^3 + \left(3\,x^2 - 1 - \alpha\right) y^2 - 3\,xy + 1 \tag{2.20}$$

Case $k_3 = 0$

This other branch of Eq.(2.15) splits into two subcases: $k_2 = 0$ and $k_2 \neq 0$. In the former case Eq.(2.15) becomes *constant invariant*, thus presenting no interest. When $k_2 \neq 0$, by introducing a new parameter α by means of

$$\alpha = k_2 k_0 - \frac{{k_1}^2}{4} \tag{2.21}$$

and changing variables in Eq.(2.15) according to

$$\left\{x = \frac{k_1 - 2\,t}{2\,k_2},\ y = \frac{2\,t - k_1}{2\,k_2} - \frac{1}{k_2 u}\right\} \tag{2.22}$$

one obtains a simpler representative for this class, depending on just one parameter α:

$$y' = \left(\alpha + x^2\right) y^3 - (2\,x + 1)\, y^2 + y \tag{2.23}$$

2.3 Generating new integrable classes from solvable members of AIA

The motivation for this work was to try to generalize into one the integrable Abel ODE classes we have seen in the literature. This goal was partially accomplished with the formulation of the AIL and AIR Abel classes, but there were still other *integrable* classes, not included in AIR, which however all had the following property: they had representatives which could be obtained from *other* Abel ODEs by changing variables $\{x \leftrightarrow y\}$. In this sense, these representatives are both of Abel and *inverse*-Abel types, which led us to ask the following question:

Which Abel ODE classes lead to other Abel classes by applying the inverse transformation $\{x \leftrightarrow y\}$ to one of its representatives?

We formulated the answer to this question in terms of the following proposition and its corollary.

Proposition If one Abel ODE, α, maps into another Abel ODE, β, by means of the inverse transformation $\{x \leftrightarrow y\}$, then both are of the *form* of Eq.(2.8).

Proof By hypothesis β is both of Abel and inverse-Abel type, that is, it is of the form $y' = G(x, y)$, where G is both cubic over linear in y (the Abel condition) *and* linear over cubic in x (the inverse-Abel condition). Hence, G is a rational function of x *and* y, with numerator cubic in y and linear in x, and denominator cubic in x and linear in y, and so β is in fact of the form Eq.(2.8). □

Corollary A given Abel ODE *class* is related through the inverse transformation to another Abel *class* if and only if it has a representative of the form of Eq.(2.8).

One consequence of this proposition is that AIA - Eq.(2.8) is in fact the *most general* ODE which is both Abel and inverse-Abel; similarly AIR - Eq.(2.7) is the most general ODE which is both Abel and inverse-Riccati, and AIL - Eq.(2.4) is the most general ODE which is both Abel and inverse-linear.

It is worth mentioning here that an Abel class may have many different representatives of the form of Eq.(2.8). Consequently, for instance, the AIL integrable class represented by Eq.(2.15), which naturally maps into a linear ODE by means of $\{x \leftrightarrow y\}$, also contains members which map into non-trivial Abel ODEs by means of the same transformation. As an example of this, consider the ODE presented in Kamke's book with the number 151:

$$y' = \frac{1 - 2\,xy + y^2 - 2\,y^3 x}{x^2 + 1} \tag{2.24}$$

This ODE has the form of Eq.(2.8) and by changing variables $\{x \leftrightarrow y\}$ it leads to a non-trivial new Abel class and nevertheless it belongs to the AIL class (see Eq.(3.12)). In other words, the class represented by this ODE has a representative of the form Eq.(2.8) other than Eq.(2.24), which by means of $\{x \leftrightarrow y\}$ maps into a linear ODE. An example where the same thing happens with a member of the AIR class is given by

$$y' = -2\,xy^2 + y^3 \tag{2.25}$$

This ODE was presented by Liouville [7], it is of the form Eq.(2.8) and by means of $\{x \leftrightarrow y\}$ leads to another non-trivial Abel class. In addition Eq.(2.25) belongs to the AIR class, so that it has another representative of the form Eq.(2.8), which by means of $\{x \leftrightarrow y\}$ maps into a Riccati ODE. Some other examples illustrating how new solvable

Table 1. *Classification of the integrable classes collected in [3].*

	Subclass AIL	Subclass AIR	Class AIA
Classes	A, C, 4, 5	B, D, 2	1, 3, 6, 7

classes can be obtained from solvable members of AIA by changing variables $\{x \leftrightarrow y\}$ are shown in the next section.

3 AIA: a generalization of known integrable classes

As mentioned in the Introduction, all the solvable classes collected in [3] - 4 depending on one parameter, labelled A, B, C and D, and another 7 not depending on parameters, labelled 1 to 7 - are particular members of the class represented by Eq.(2.8). The "classification" of these solvable classes as particular members of AIL, AIR or AIA is as follows:

Although this classification shown in Table 1 is not difficult to verify, we think it worthwhile to show it explicitly. Starting with the parameterized class by Abel [1], shown in [3] as "Class A",

$$y' = \left(\alpha\, x + \frac{1}{x} + \frac{1}{x^3}\right) y^3 + y^2 \tag{3.1}$$

where α is the parameter, this ODE can be obtained from Eq.(2.16) by taking $\{k_3 = 2\,\alpha, k_2 = -1, k_1 = 1/2, k_0 = 0\}$ and changing variables $\{x = \frac{t^2}{2}, y = 2\,\frac{u+t}{t^3}\}$. So Eq.(3.1) is member of AIL.

Concerning the 1-parameter class by Liouville [8], labelled in [3] as class "B",

$$y' = 2\,\left(x^2 - \alpha\right) y^3 + 2\,\left(x + 1\right) y^2 \tag{3.2}$$

this ODE is obtained from Eq.(2.7) by taking $\{s_2 = 0, s_1 = 0, s_0 = 1, r_2 = 1, r_1 = 0, r_0 = 0, a_3 = 0, a_2 = 0, a_0 = -2\,\alpha, a_1 = -2\}$ and changing variables $\{x = t, y = \frac{-1}{u} - t^2\}$. Eq.(3.2) is then member of AIR.

The next solvable class, related to Abel's work, presented in [3] as class "C", is given by

$$y' = \frac{\alpha\left(1 - x^2\right) y^3}{2\,x} + \left(\alpha - 1\right) y^2 - \frac{\alpha y}{2\,x} \tag{3.3}$$

and it can be obtained from Eq.(2.16) by taking $\{k_2 = 0,\ k_1 = \alpha/2,\ k_0 = 0\ k_3 = -1/2\}$ and changing variables $\{x = \frac{\sqrt{\alpha}}{t}, y = \frac{t\,(1-t\,u)}{\sqrt{\alpha}}\}$; Eq.(3.3) is then member of AIL.

The last *parameterized* solvable class shown in [3], labelled there as class "D", is related to Appell's work [2], and is given by

$$y' = -\frac{y^3}{x} - \frac{\left(\alpha + x^2\right) y^2}{x^2} \tag{3.4}$$

This ODE is obtained from Eq.(2.7) by taking $\{s_2 = 0, s_1 = 1, s_0 = 0, r_2 = 1, r_1 = 0, r_0 =$

$-\alpha, a_3 = 0, a_2 = 0, a_1 = 0, a_0 = -1\}$ and changing variables $\{x = t, y = \frac{1}{u} + \frac{\alpha - t^2}{t}\}$. So, Eq.(3.4) is member of AIR.

Concerning the classes collected in [3] not depending on parameters, the first one, there labelled as "Class 1", was presented by Halphen [5] in connection with doubly periodic elliptic functions:

$$y' = \frac{3\,y\,(1+y) - 4\,x}{x\,(8\,y-1)} \tag{3.5}$$

This ODE, clearly a member of Eq.(2.8), can be obtained by changing $\{x \leftrightarrow y\}$ in $y' = \frac{y(8\,x-1)}{3\,x(x+1)-4\,y}$, which in turn can be obtained from Eq.(3.3) (the solvable AIL class) by taking $\alpha = -2/3$ and changing variables $\{y = \frac{27\,t^2\,(2\,t-1)}{(t+1)(3\,t^2+3\,t-4\,u(t))\sqrt{3-6\,t}} - \frac{9\,t\,(2\,t-1)}{(t+1)^2\sqrt{3-6\,t}}, x = \frac{(1-2\,t)\sqrt{3-6\,t}}{9\,t}\}$. This derivation also illustrates how new solvable classes can be obtained by interchanging the roles between dependent and independent variables in *solvable* members of AIA.

As for the representative of Class 2, by Liouville [7], shown in [3] as

$$y' = y^3 - 2\,x\,y^2 \tag{3.6}$$

this ODE is obtained from Eq.(2.7) by taking $\{s_2 = 0, s_1 = 0, s_0 = 1, r_2 = 1, r_1 = 0, r_0 = 0, a_3 = 0, a_2 = 0, a_1 = 0, a_0 = 1\}$ and converting the resulting equation into first kind format by changing variables $\{x = t, y = \frac{1}{u} - t^2\}$ - Eq.(3.6) is then member of AIR.

Also of note here, Eq.(3.6) is a special case of the ODE presented by Appell in [2]:

$$y' = -\frac{y^3}{\alpha\,x^2 + \beta\,x + \gamma} - \frac{d}{dx}\left(\frac{ax^2+bx+c}{\alpha\,x^2+\beta\,x+\gamma}\right) y^2 \tag{3.7}$$

with $\alpha = 0, \beta = 0, \gamma = -1, a = 1, b = 0, c = 0$. Eq.(3.7) is also seen to be a member of AIR since it can be obtained by changing $\{x = t, y = \frac{1}{u} - \frac{at^2+bt+c}{\alpha\,t^2+\beta\,t+\gamma}\}$ in Eq.(2.7) with $s_2 = \alpha, s_1 = \beta, s_0 = \gamma, r_2 = a, r_1 = b, r_0 = c, a_3 = 0, a_2 = 0, a_1 = 0, a_0 = -1$.

Class 3, also by Liouville [8], presented in [3] as

$$y' = \frac{y^3}{4\,x^2} - y^2 \tag{3.8}$$

can be obtained from Eq.(3.6) by changing $\{x \leftrightarrow y\}$ and then converting it to first kind format by means of $\{x = 2\,t, y = \frac{-1}{u} + t\}$ - Eq.(3.8) is then member of AIA. This also illustrates the derivation of a solvable class by changing $\{x \leftrightarrow y\}$ in *solvable* members of the AIR subclass.

The next class, presented in [3] as Class 4, collected among the Abel ODE examples of Kamke's book,

$$y' = y^3 - \frac{(x+1)\,y^2}{x} \tag{3.9}$$

can be obtained from Eq.(2.23) by taking $\alpha = 0$ and changing variables $\{x = \frac{-1}{t}, y = t^2 u - t\}$; so it belongs to the AIL subclass.

In [3], three new integrable classes not depending on parameters were presented too - these are classes "5", "6" and "7". Starting with class 5, given by

$$y' = -\frac{(2\,x+3)\,(x+1)\,y^3}{2\,x^5} + \frac{(5\,x+8)\,y^2}{2\,x^3} \tag{3.10}$$

this ODE can be obtained from Eq.(2.4) by taking $\{s_1 = 0, s_0 = 1, r_1 = 1, r_0 = 0, a_3 = -6, a_2 = 10, a_1 = -4, a_0 = 0\}$ and changing variables $\{x = \frac{1}{t}, y = \frac{2\,t^2-(t+1)u}{t((t+1)u+2\,t)}\}$.

Regarding Class 6, given by

$$y' = -\frac{y^3}{x^2\,(x-1)^2} + \frac{\left(1-x-x^2\right)y^2}{x^2\,(x-1)^2} \tag{3.11}$$

this ODE is obtained from Eq.(3.9) by changing $\{x \leftrightarrow y\}$ and then converting it to first kind format by means of $\{x = t, y = \frac{t-1+u}{(t-1)u}\}$.

Concluding with these new classes presented in [3], for class 7, a more convenient representative, free of radicals, is obtained by changing variables in the representative shown in [3] according to $\{x = t, y = 2\sqrt{t}\left(t^2+1\right)^{5/4}u\}$, leading to the ODE with rational coefficients

$$y' = \frac{\left(4\,x^2+1\right)\left(x^2+1\right)y^3}{2\,x} - xy^2 - \frac{\left(6\,x^2+1\right)y}{2\,x\,(x^2+1)} \tag{3.12}$$

This ODE can be obtained from Kamke's first order example 151,

$$y' = \frac{\left(y^2+1\right)(1-2\,yx)}{x^2+1} \tag{3.13}$$

by changing $\{x \leftrightarrow y\}$ and then converting the resulting ODE to first kind format by means of $\{x = t, y = \frac{1}{2\,t} + \frac{1}{2\,t\,(t^2+1)\,u}\}$. In turn, Eq.(3.13) is member of the AIL subclass and can be obtained from Eq.(3.3) by taking $\alpha = -4$ and changing variables $\{x = \frac{i}{t}, y = \frac{i\,t\,(t\,u-1)}{t^2+1}\}$.

4 Conclusions

In this paper, a multi-parameter non-constant invariant Abel class was presented which generalizes in one the integrable cases shown in the works of the 19^{th} century by Abel, Liouville, Appell and others, including all those shown in Kamke's book and other references. This new class splits into various subclasses, many of which are fully integrable, including some not previously shown elsewhere to the best of our knowledge.

The particular subclass represented by Eq.(2.15) mapping non-constant invariant Abel ODEs into linear first order ODEs contains by itself most of the exactly integrable cases we have seen. The subclass represented by Eq.(2.17) appears to us to be the most general class mapping Abel ODEs into Riccati ones; indeed it includes the parameterized mappings presented by Liouville and Appell as particular members.

Finally the mapping of Abel classes into other Abel classes presents a useful way of finding new integrable classes from other classes known to be solvable, as shown in sec. 3.

We are presently working in analyzing different connections between all these subclasses and expect to find reportable results in the near future.

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