On the Class Field of the Quadratic Number Field with Discriminant -47

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February 1964

Translated from German to English by Steven Kieffer

January 2010

Among imaginary quadratic number fields $\Omega = \mathbb{Q}(\sqrt{d})$ with prime discriminant $d = -p\,(\equiv 1 \mod 4)$ and odd class number h, the first cases with $h > 1$ are well known:

$$\begin{array}{ll} d = -23 & h = 3, \\ d = -31 & h = 3, \\ d = -47 & h = 5. \end{array}$$

In general the class field N of Ω is normal over the rational number field $\mathbb{Q}$, with dihedral Galois group of order $2h$. The cyclic normal subgroup of order h corresponds to the quadratic subfield Ω. For the h conjugate subgroups of order 2, the corresponding subfields K are extensions of degree h over $\mathbb{Q}$, and one of these is characterized as the maximal real subfield of N.

Whereas it is easy in the cases $d = -23$ and $d = -31$ to give an explicit arithmetic-canonical representation of the so defined number field K of degree $h = 3$, the same problem in the case $d = -47$, where K has degree 5, has until now been solved only through equations flowing from the transformation theory of modular functions (i.e. so-called modular equations), so that the arithmetical nature of the roots in the class field N remains in the dark. [1]

Some time ago H. Koch (Berlin) asked me whether one could not handle the case of $d = -47$ just as simply as the cases $d = -23$ and $d = -31$,

[1]See H. Weber, *Algebra III*, 2nd ed., Braunschweig 1908, §131, as well as R. Fricke,

and at first I thought that indeed one easily could. On closer inspection it became apparent however, that for $d = -47$ a considerably greater effort would be necessary. I communicate hereafter the result of my efforts, in whose final stage I was aided by Klaus Alber (Hamburg). [2] In this I rely on the arithmetic theory of cyclic biquadratic number fields, as I have developed it in an earlier work, [3] as well as certain results from my monograph on the class number of Abelian number fields. [4] I begin by clarifying, by means of the examples $d = -23$ and $d = -31$, what kind of arithmetic-canonical representation I have in mind.

1 The Cases $d = -23$ and $d = -31$

The two cases allow for a common treatment; they differ only by the sign in

$$d = -27 \pm 4 = -3^3 \pm 2^2.$$

In the following, the upper and lower signs always correspond, respectively, to the cases $d = -23$ and $d = -31$.

1.1 Ascent to Generation of N^3/Ω^3

The cyclic extension of 3rd degree N/Ω can be generated by a radical, after the adjunction of a third root of unity (indicated by superscript 3):

$$N^3 = \Omega^3 \left(\sqrt[3]{\omega}\right).$$

Algebra III, Braunschweig 1928, §4 (p. 492). There the equations

$$x^5 - x^3 - 2x^2 - 2x - 1 = 0$$

resp.

$$x^5 - x^4 + x^3 + x^2 - 2x_{+1} = 0$$

are given as resolvents of the class equation for the discriminant $d = -47$.

[2] Hasse's student, PhD Hamburg 1959, dissertation: *Einige Sätze aus der komplexen Multiplikation.*

[3] H. Hasse, *Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörpern*, Abh. Deutsche Akad. Wiss. Berlin 1948, Nr. 2 (1950). Cited in the following by GK.

[4] H. Hasse, *Über die Klassenzahl abelscher Zahlkörper*, Akad. Verlag, Berlin 1952. Cited in the following by KAZ.

To be precise, the radicand ω is a singular 3-primary number of Ω^3, that is, a third divisor power number [5] which is 3-primary (i.e. a third power residue mod $3\sqrt{-3}$). In short, [6]

$$\omega \underset{3}{\cong} 1, \quad \omega \underset{3}{\equiv} 1 \quad \mathrm{mod}\ 3\sqrt{-3}.$$

In the sense of $\underset{3}{=}$, it is uniquely determined, up to choice of $\omega^{\pm 1}$ (i.e. up to third number power factors [7]).

In order to investigate ω, we will first determine the fundamental unit ε and class number h of the bicyclic biquadratic number field

$$\Omega^3 = \mathbb{Q}(\sqrt{-3}, \sqrt{d}).$$

That can be done, following KAZ §26, in the following way.

The real quadratic subfield

$$\Omega_0^3 = \mathbb{Q}(\sqrt{-3d})$$

has the fundamental unit

$$\varepsilon_0 = \frac{(27 \mp 2) + 3\sqrt{-3d}}{2}$$

with the norm

$$N(\varepsilon_0) = 1$$

and the class number

$$h_0 = 1.$$

The two imaginary quadratic subfields

$$\Omega = \mathbb{Q}(\sqrt{d}), \quad \mathbb{Q}^3 = \mathbb{Q}(\sqrt{-3})$$

have the class numbers

$$h_1 = 3, \quad h_2 = 1.$$

[5] That is, a number whose divisor is a third power.

[6] Hasse's notation in which a number n is set beneath an equivalence relation $=$, $\cong$, or $\equiv$ means that the relation holds up to an n^{th} power factor. That is, there exists a number or a divisor such that the relation will hold if one side is multiplied by its n^{th} power. The relation $\alpha \cong \beta$ means that α and β have the same divisor, if α, β are numbers; if β is a divisor then it means that β is the divisor of α.

[7] That is, up to a factor which is the third power of a number.

Following KAZ, §26, (6) - (8), one has in general that

$$\varepsilon = \begin{cases} \varepsilon_0 & \text{if} \quad Q = 1 \\ \sqrt{-\varepsilon_0} & \text{if} \quad Q = 2 \end{cases}$$

and

$$h_1 = \frac{1}{2} Q h_0 h_1 h_2, \quad h^* = \frac{1}{2} Q h_1 h_2,$$

where Q is the unit index [8] and h^* is the relative class number of Ω^3/Ω_0^3. The unit index is by definition

$$Q = \left\{ \begin{array}{ll} 1 & \text{if} - \varepsilon_0 \underset{2}{\neq} 1 \\ 2 & \text{if} - \varepsilon_0 \underset{2}{=} 1 \end{array} \right\} \quad \text{in} \quad \Omega^3.$$

Already, since the relative class number is integral (KAZ §§19, 27), we must have $Q = 2$ in the present case, where $h^* = \frac{1}{2} Q \cdot 3$. One can see this in the following way, without having to appeal to the rather deep-lying integrality of h, and thereby at the same time determine ε.

The criterion (11_I) for $Q = 2$ from KAZ, §26, (12_I) is satisfied:

$$\mp \sqrt{-3}^2 = \pm 3 = a a'$$

with

$$a = \frac{9 + \sqrt{-3d}}{2}$$

from Ω_0^3. Since the prime 3 is ramified in Ω_0^3, one therefore has

$$a \cong a', \quad \text{so that} \quad a \cong \sqrt{-3}.$$

Therefore

$$\varepsilon = \frac{a}{\sqrt{-3}} = \frac{-3\sqrt{-3} + \sqrt{d}}{2}$$

is a unit in Ω^3, with

$$\varepsilon^2 = \frac{a^2}{-3} = \frac{(-27 + d)/2 - 3\sqrt{-3d}}{2} = \frac{-(27 \mp 2) - 3\sqrt{-3d}}{2} = -\varepsilon_0.$$

[8]The terms *unit index*, and *relative class number* are defined in Hasse's monograph (Hasse 1952).

But this says that $Q=2$ and ε is the fundamental unit of Ω^3. Therewith we get that

$$h = h^* = 3.$$

From class field theory it follows that in Ω^3 there is essentially only one singular 3-primary number ω. The two postulates on ω are now satisfied for the fundamental unit ε, the first trivially, since actually $\varepsilon \cong 1$; the second considering that

$$\varepsilon^{-1} \underset{3}{=} \varepsilon^2 = -\varepsilon_0 \equiv \frac{d \mp 2}{2} \equiv \pm 1 \underset{3}{\equiv} 1 \mod 3\sqrt{-3}.$$

Then one can normalize by setting

$$\omega = \varepsilon$$

so that we get for N^3/Ω^3 the representation

$$N^3 = \Omega^3\left(\sqrt[3]{\varepsilon}\right).$$

Remark: One notes that in the present cases $d=-23$ and $d=-31$, the singular 3-primary number ω is not, as one would have assumed at the outset, formed by normalization of the third divisor power number already existing in Ω,

$$w = \frac{(2 \pm 1) + \sqrt{d}}{2} \underset{3}{\cong} 1 \quad \text{with} \quad N(w) = 2^3$$

by means of a unit of Ω^3. In the following treatment, the case $d=-47$ will turn out accordingly.

1.2 Descent to Generation of $K/\mathbb{Q}$

For the present purpose it is crucial that the singular 3-primary number ω in Ω^3 can be normalized as a number already in the maximal real subfield Ω_0^3 of Ω^3. That is achieved either by the normalization

$$\omega^{-1} \underset{3}{=} \varepsilon^2 \underset{3}{=} \varepsilon_0 = \frac{(27 \mp 2) + 3\sqrt{-3d}}{2}$$

just now given, or better, since – as will be seen – it leads to a lower fundamental equation, [9] the normalization

$$\omega \underset{3}{=} \frac{\varepsilon}{\sqrt{-3}^3} = \frac{\alpha}{9} = \frac{1}{9} \cdot \frac{9+\sqrt{-3d}}{2}.$$

Since the (real) radical $\sqrt[3]{\alpha/9}$ already lies in the maximal real subfield N_0^3 of N^3, one therefore has for N_0^3/Ω_0^3 the representation

$$N_0^3 = \Omega_0^3\left(\sqrt[3]{\frac{\alpha}{9}}\right).$$

The generating automorphism of N_0^3/K sends the radical $\sqrt[3]{\alpha/9}$ to $\sqrt[3]{\alpha'/9}$ (again meaning the real root). Therefore the trace of N_0^3/K in K is the radical sum

$$A = \sqrt[3]{\frac{\alpha}{9}} + \sqrt[3]{\frac{\alpha'}{9}}$$

and this generates K over $\mathbb{Q}$, since it is different from both its complex conjugates. For the generator A one has

$$A^3 = \left(\frac{\alpha}{9}+\frac{\alpha'}{9}\right) + 3\sqrt[3]{\frac{\alpha}{9}}\sqrt[3]{\frac{\alpha'}{9}}\left(\sqrt[3]{\frac{\alpha}{9}}+\sqrt[3]{\frac{\alpha'}{9}}\right).$$

Considering that $\alpha+\alpha' = 9$ and $\alpha\alpha' = \pm 3$, this becomes

$$A^3 = 1 \pm A.$$

Therewith the stated goal is reached:

Result. *The maximal real subfield K of the class field N of an imaginary quadratic number field $\Omega = \mathbb{Q}(\sqrt{d})$ with $d = -27 \pm 4$ has the representation*

$$K = \mathbb{Q}(A)$$

with the minimal equation

$$A^3 \mp A - 1 = 0.$$

[9] By a fundamental equation for a field, Hasse means the equation $f(x) = 0$ in which $f(x)$ is the minimal polynomial of a primitive or generating element of the field. By lower he means lower height, i.e. smaller coefficients.

The generator A is arithmetically characterized as the radical sum

$$A = \sqrt[3]{\frac{\alpha}{9}} + \sqrt[3]{\frac{\alpha'}{9}},$$

whose radicand

$$\frac{\alpha}{9} = \frac{1}{9} \cdot \frac{9 + \sqrt{-3d}}{2}$$

is related to the fundamental unit

$$\varepsilon_0 = \frac{(27 \mp 2) + 3\sqrt{-3d}}{2}$$

of the maximal real subfield $\Omega_0^3 = \mathbb{Q}(\sqrt{-3d})$ of $\Omega^3 = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$, and to the prime number 3 by

$$\frac{\alpha}{\alpha'} = \pm\varepsilon_0, \quad \alpha\alpha' = \pm 3.$$

Remark. Using the aforementioned, more obvious normalization $\omega^{-1} \underset{3}{=} \varepsilon_0$ of the radicands we get for the generator

$$B = \sqrt[3]{\varepsilon_0} + \sqrt[3]{\varepsilon_0'}$$

the higher fundamental equation

$$B^3 - 3B - (27 \mp 2) = 0.$$

2 The Case $d = -47$

For the sake of uniformity with the afore-handled cases, although here d has only the one value -47, the notation $\sqrt{d}$ will be retained (and we will not write $\sqrt{-47}$).

2.1 Ascent to Generation of N^5/Ω^5

The cyclic extension of 5th degree N/Ω can be generated by a radical, after the adjunction of a fifth root of unity (indicated by superscript 5):

$$N^5 = \Omega^5 \left(\sqrt[5]{\omega}\right),$$

where the radicand ω is a singular 5-primary number of Ω^5, that is, [10] a 5th divisor power number, which is 5-primary (i.e. a fifth power residue mod $5\sqrt{-e\sqrt{5}}$). In short,

$$\omega \underset{5}{\cong} 1, \quad \omega \underset{5}{\equiv} 1 \mod 5\sqrt{-e\sqrt{5}}.$$

Up to choice of $\omega^{\pm 1}$, $\omega^{\pm 2}$, it is uniquely determined in the sense of $\underset{5}{=}$ (i.e. up to 5th number power factors).

In order to investigate ω, we will first, by the method laid out in full in GK, determine the relative fundamental unit ε_0, unit index Q_0 and class number h_0 of the cyclic biquadratic maximal real subfield

$$\Omega_0^5 = \mathbb{Q}\left(\sqrt{-e\sqrt{5}\cdot d}\right)$$

of Ω^5 and thence the unit index Q, and furthermore the fundamental unit ε and class number h of the imaginary Abelian number field

$$\Omega^5 = \mathbb{Q}\left(\sqrt{-e\sqrt{5}}, \sqrt{d}\right)$$

of Type $(4,2)$.

2.1.1 Relative fundamental unit ε_0, unit index Q_0 and class number h_0 of Ω_0^5

The determination is based on the representation of the integers of Ω_0^5 in the canonical form

$$\frac{1}{2}\left(\frac{x_0 + x_1\tau(\psi)}{2} + y_0\frac{\tau(\chi)+\tau(\overline{\chi})}{2} + y_1\frac{i\tau(\chi)-i\tau(\overline{\chi})}{2}\right)$$

with congruence conditions mod 4 for the rational integer coordinates x_0, x_1, y_0, y_1 (GK, §8, (2), (3) and Theorem 14). Here $\chi, \overline{\chi}$ are the two conjugate biquadratic characters mod $5\cdot 47$, $\psi = \chi^2$ the quadratic character mod 5, and

[10] We note that in a diagram that we have not reproduced in this translation, Hasse defines $e = (1+\sqrt{5})/2$, noting that it is the fundamental unit of $\mathbb{Q}_0^5$.

$\tau(\chi), \tau(\overline{\chi}), \tau(\psi)$ the corresponding Gauss sums. The forms appearing in this representation are given, following GK, §8, (21), by

$$\tau(\psi) = \sqrt{5}, \quad \frac{\tau(\chi)+\tau(\overline{\chi})}{2} = e'\sqrt{-e\sqrt{5}\cdot d}, \quad \frac{i\tau(\chi)-i\tau(\overline{\chi})}{2} = \sqrt{-e\sqrt{5}\cdot d}.$$

The relative units of $\Omega_0^5/\mathbb{Q}_0^5$ (units with relative norm ± 1) are characterized in this canonical representation, following GK, §12, (2), by the coordinate equations

$$\frac{(x_0^2 \mp 16)/5 + x_1^2}{2\cdot 47} = y_0^2 + y_1^2, x_0x_1 = -(y_0^2 - y_1^2) - 4y_0y_1,$$

and the relative unit ε_0 corresponds to the essentially uniquely determined solution with minimal $y_0^2 + y_1^2$. As minimal solution one quickly finds here, using the systematic testing procedure in GK, §12, A1,

$$x_0 = 47, \quad x_1 = -5, \quad y_0 = -1, \quad y_1 = -2,$$

with positive sign in the first equation on the left, which means relative norm -1; y_0, y_1 are thereby negatively normalized, for entirely irrelevant reasons, not to be discussed here. Therefore $\Omega_0^5/\mathbb{Q}_0^5$ has the relative fundamental unit

$$\varepsilon_0 = \frac{1}{2}\left(\frac{47-5\sqrt{5}}{2} - e'\sqrt{-e\sqrt{5}\cdot d} - 2\sqrt{-e\sqrt{5}\cdot d}\right)$$

or

$$\varepsilon_0 = \frac{1}{2}\left(\frac{47-5\sqrt{5}}{2} - \frac{-5+\sqrt{5}}{2}\sqrt{-e\sqrt{5}\cdot d}\right)$$

with the relative norm

$$n(\varepsilon_0) = \varepsilon_0\varepsilon_0'' = -1.$$

Since $x_1 = -5$ thus proves not to be divisible by 47, we finally get, using GK, §12, A2, the unit index of $\Omega_0^5/\mathbb{Q}_0^5$ as

$$Q_0 = 1.$$

Accordingly, the unit group of Ω_0^5 will be generated by the units

$$e, \varepsilon_0, \varepsilon_0' \quad \text{with} \quad N(e) = -1, n(\varepsilon_0) = -1.$$

In order to at last determine the class number h_0 of Ω_0^5, one has to compute the reduced relative cyclotomic unit [11]

$$\eta_0 = \theta\theta'$$

(GK, §19, (11)). That can be done using the Bergström product formula for the 23-factor product

$$\theta = \prod_a \left((-\zeta)^a - (-\zeta)^{-a}\right)$$

(GK, §14), where ζ is a primitive $5 \cdot 47$-th root of unity, and a runs over an odd normalized subsystem of the rational congruence group mod $5 \cdot 47$ assigned to Ω_0^5 (KAZ, §10), say, the smallest positive

$$1, 19, 21, 29, 39, 51, 61, 69, 71, 81, 99, 101, 109,$$

$$111, 121, 129, 131, 139, 179, 191, 199, 219, 229.$$

For this calculation one has to turn to the Schema in GK, §19, B 3, a 2, Type 3 (p. 86). Without mechanical or electronic computational means this could not have been carried out successfully. By means of the Hamburg electronic computer TR 4 we obtained the agreement

$$\eta_0 = \frac{1}{2}\left(\frac{47 - 5\sqrt{5}}{2} - \frac{-5 + \sqrt{5}}{2}\sqrt{-e\sqrt{5} \cdot d}\right) = \varepsilon_0$$

with the afore-determined relative fundamental unit ε_0. Following GK, §19, Theorem 37 implies, in light of $Q_0 = 1$, that $\Omega_0^5/\mathbb{Q}_0^5$ has the relative class number

$$h_0^* = 1.$$

Since $\mathbb{Q}_0^5 = \mathbb{Q}(\sqrt{5})$ has the class number 1, therefore Ω_0^5 also has the class number

$$h_0 = 1.$$

[11] On cyclotomic units, see e.g. (Hilbert 1897, §98).

2.1.2 Unit index Q, another fundamental unit ε, and class number h of Ω^5

By KAZ, §33, (p. 98) Ω^5/Ω^5_0 has the unit index

$$Q = 2.$$

A unit ε therefore existing in Ω^5 by KAZ, §20, Theorem 14 and ($4\mathrm{a}_0$), with the property

$$\overline{\varepsilon} = -\varepsilon,$$

will arise by the generating automorphism of Ω^5/Ω^5_0 (complex conjugation), from the number θ underlying the reduced relative cyclotomic unit $\eta_0 = \theta\theta'$ – and generally all cyclotomic units – which indeed in the present case (a composite conductor $f = 5 \cdot 47$) is a unit in Ω^5, and as product of an odd number of pure-imaginary factors with the property

$$\overline{\theta} = -\theta$$

satisfies:

$$\varepsilon = \theta.$$

For this number θ we found during the aforementioned electronic computation of η_0 the value

$$\theta = \frac{1}{2}\left((2-\sqrt{5})\sqrt{d} + \frac{25 - 11\sqrt{5}}{2}\sqrt{-e\sqrt{5}}\right).$$

As one also easily computes, the relative norm of this unit of $\mathbb{Q}(\sqrt{5}, \sqrt{d})$ is

$$n(\theta) = \theta\theta' = e' = -e^{-1}$$

and therefore its complete norm is

$$N(\theta) = \theta\theta'\theta''\theta''' = N(e) = -1.$$

By the salient points demonstrated thus far, the unit group of Ω^5 must be generated by a primitive $5 \cdot 47$th root of unity ζ, the real units e, ε_0, ε_0', and the imaginary unit θ. According to the norm relations just given and the earlier relation

$$\varepsilon_0 = \eta_0 = \theta\theta'$$

one can instead simply take

$$\zeta \quad \text{and the conjugates} \quad \theta, \theta', \theta''$$

as generators.

By KAZ, §33, Theorem 34, we compute finally, in light of $Q = 2$, the relative class number of Ω^5/Ω_0^5 as

$$h^* = 2 \cdot 10 \cdot N_\chi(\theta(\chi)) N_\psi(\theta(\psi)) N_{\hat{\psi}}(\theta(\hat{\psi})),$$

where, as earlier, χ denotes a biquadratic character mod $5 \cdot 47$, $\psi = \chi^2$ the quadratic character mod 5, and $\hat{\psi}$ the quadratic character mod 47, and $\theta(\chi)$, $\theta(\psi)$, $\theta(\hat{\psi})$ are the character sums formed according to KAZ, §27, (2), of which the norms N_χ, N_ψ, $N_{\hat{\psi}}$ are taken in the field of the respective characters. The calculation of these character sums and their norms can without great effort be carried out by hand. They give as relative class number

$$h^* = 2 \cdot 5.$$

In view of $h_0 = 1$, Ω^5 therefore also has the class number

$$h = 2 \cdot 5.$$

Therefore by class field theory there is in Ω^5 essentially only *one* singular 5-primary number ω.

2.1.3 Determination of the singular 5-primary number ω

The singular 5-primary number ω we seek must comprise units of Ω^5, and the essentially unique 5th divisor power number already existing in Ω,

$$w = \frac{9 + \sqrt{d}}{2} \underset{5}{\cong} 1 \quad \text{with} \quad N(w) = 2^5$$

In order to reach it by a suitable product of powers of w and the fundamental units ζ, θ, θ', θ'' of Ω^5, one determines the exponents mod 5 in the representation of these numbers by a basis of the π-adic principal unit group [12] of Ω^5, only considered mod π^5, where for short we write

$$\pi = \sqrt{-e\sqrt{5}} \quad \text{with} \quad \pi^4 \cong 5,\ \pi^5 \cong 5\pi$$

[12] See (Hensel 1908, Ch. 4 § 7).

for the prime divisor of 5 in Ω^5. This analysis can be made without great difficulty, by an incremental procedure, through rising powers $\pi, \pi^2, \pi^3, \pi^4, \pi^5$ as modulus. The result is assembled in the following table, in whose heading the chosen basis is given; blank spaces indicate exponents 0 mod 5:

	$1+\pi$	$1+\pi\sqrt{d}$	$1+\sqrt{5}$	$1+\sqrt{5}\sqrt{d}$	$1+\pi\sqrt{5}$	$1+\pi\sqrt{5}\sqrt{d}$	$1+5$	$1+5\sqrt{d}$	
w								3	
ζ	3		3		4		4		
θ			2			2	1		2
θ'			3			1	4		3
θ''			2			3	1		1

Between the five exponent lines there is clearly, as there must be, a linear dependence mod 5, namely with the coefficients given in the column on the far right. Therefore the unit

$$\omega = \theta^2\theta'^3\theta'' \underset{5}{\equiv} 1 \mod 5\pi,$$

is 5-primary. By carrying out the formation of conjugates and the multiplication, which is easiest in the association

$$\omega = (\theta\theta')^2(\theta'\theta'') = \eta_0^2\eta_0' = \varepsilon_0^2\varepsilon_0',$$

one obtains for ω the value

$$\omega = \frac{1}{2}\left(\frac{9353 + 4225\sqrt{5}}{2} - \frac{715 + 325\sqrt{5}}{2}\sqrt{-e\sqrt{5}\cdot d}\right).$$

For use later let us also note the value

$$\varepsilon_0\varepsilon_0' = \frac{1}{2}\left(\frac{521 + 235\sqrt{5}}{2} - (20 + 9\sqrt{5})\sqrt{-e\sqrt{5}\cdot d}\right).$$

With the radicands ω so determined, one then has for N^5/Ω^5 the representation

$$N^5 = \Omega^5\left(\sqrt[5]{\omega}\right) = \Omega^5\left(\sqrt[5]{\varepsilon_0^2\varepsilon_0'}\right).$$

Remark. One notes that also in the present case $d = -47$ by construction the 5th divisor power number w in Ω does not divide the singular 5-power number, entirely analogously to the afore-handled cases $d = -23$ and $d = -31$.

2.2 Descent to Generation of $K/\mathbb{Q}$

The afore-determined singular 5-primary number $\omega = \varepsilon_0^2\varepsilon_o'$ of Ω^5 is here already so normalized that it lies in the maximal real subfield Ω_0^5 of Ω^5. Since the (real) radical $\sqrt[5]{\omega} = \sqrt[5]{\varepsilon_0^2\varepsilon_0'}$ lies in the maximal real subfield N_0^5 of N^5, one thus has for N_0^5/Ω_0^5 the representation

$$N_0^5 = \Omega_0^5(\sqrt[5]{\omega}) = \Omega_0^5(\sqrt[5]{\varepsilon_0^2\varepsilon_0'}).$$

2.2.1 Descent from N_0^5/Ω_0^5 to $K_0^5/\mathbb{Q}_0^5$.

The generating automorphism of N_0^5/K_0^5 sends the radical $\sqrt[5]{\omega}$ to $\sqrt[5]{\omega''}$ (again, the real radical understood). In view of

$$\omega\omega'' = n(\varepsilon_0)^3 = -1$$

one also has

$$\sqrt[5]{\omega}\sqrt[5]{\omega''} = -1.$$

We have then the radical sum

$$B = \sqrt[5]{\omega} + \sqrt[5]{\omega''} = \sqrt[5]{\omega} - \frac{1}{\sqrt[5]{\omega}}$$

as the trace for N_0^5/K_0^5 in the subfield K_0^5, and this generates this subfield over $\mathbb{Q}_0^5$, since it is different from its four complex conjugates:

$$K_0^5 = \mathbb{Q}_0^5(B).$$

In order to find the equation that the B in this representation satisfies, one goes to the identities

$$\left(x - \frac{1}{x}\right)^5 = \left(x^5 - \frac{1}{x^5}\right) - \left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right),$$

$$\left(x - \frac{1}{x}\right)^3 = \left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right),$$

thus

$$\left(x - \frac{1}{x}\right)^5 + 5\left(x - \frac{1}{x}\right)^3 + 5\left(x - \frac{1}{x}\right) = x^5 - \frac{1}{x^5},$$

and puts $x = \sqrt[5]{\omega}$ therein. This gives:

$$B^5 + 5B^3 + 5B = \omega - \frac{1}{\omega} = \omega + \omega'' = \frac{9353 + 4225\sqrt{5}}{2}.$$

The norm of the absolute term of this equation is the prime number 443629.

2.2.2 Descent from $K_0^5/\mathbb{Q}_0^5$ to $K/\mathbb{Q}$.

The generating automorphism of K_0^5/K sends the radical $\sqrt[5]{\omega}$ to $\sqrt[5]{\omega''}$ (again, the real radical understood). In view of

$$\omega' = \varepsilon_0'^2 \varepsilon_0'' = \frac{\varepsilon_0^5 \varepsilon_0'^2 \varepsilon_0''}{\varepsilon_0^5} = -\frac{\varepsilon_0^4 \varepsilon_0'^2}{\varepsilon_0^5} = -\frac{\omega^2}{\varepsilon_0^5}$$

one also has

$$\sqrt[5]{\omega'} = -\frac{\sqrt[5]{\omega}^2}{\varepsilon_0}.$$

We have then the radical sum

$$A = B + B' = \sqrt[5]{\omega} + \sqrt[5]{\omega'} + \sqrt[5]{\omega''} + \sqrt[5]{\omega'''} = \left(\sqrt[5]{\omega} - \frac{1}{\sqrt[5]{\omega}}\right) - \left(\frac{\sqrt[5]{\omega}^2}{\varepsilon_0} - \frac{\varepsilon_0}{\sqrt[5]{\omega}^2}\right)$$

as the trace for K_0^5/K in the subfield K, and this generates this subfield over $\mathbb{Q}$, since it is different from its four complex conjugates:

$$K = \mathbb{Q}(A).$$

In order to finally find the equation which the A in this representation satisfies, one may regard the last given representation of A as a representation by the basis $\sqrt[5]{\omega}^{-2}, \sqrt[5]{\omega}^{-1}, 1, \sqrt[5]{\omega}, \sqrt[5]{\omega}^2$ of N_0^5/Ω^5, and compute from that the corresponding basis representations of A^2, A^3, A^4, and A^5. For carrying out this computation, which was indeed somewhat troublesome, yet to be dealt with entirely by hand, I thank Klaus Alber (Hamburg). The equation obtained was

$$A^5 + 10A^3 - 5T(\varepsilon_0)A^2 + 5\left(1 + T\left(\frac{\omega}{\varepsilon_0}\right)\right)A - T(\omega) = 0.$$

One computes the traces appearing therein from the afore-given numerical values of ε_0, $\omega/\varepsilon_0 = \varepsilon_0\varepsilon_0'$, $\omega = \varepsilon_0^2\varepsilon_0'$ to be

$$T(\varepsilon_0) = 47, \quad T\left(\frac{\omega}{\varepsilon_0}\right) = 521, \quad T(\omega) = 9353.$$

Therewith is the stated goal reached:

Result. *The maximal real subfield K of the class field N of the imaginary quadratic number field $\Omega = \mathbb{Q}(\sqrt{-47})$ has the representation*

$$K = \mathbb{Q}(A)$$

with the fundamental equation

$$A^5 + 10A^3 - 235A^2 + 2610A - 9353 = 0.$$

The generator A is arithmetically characterized as the radical sum

$$A = \sqrt[5]{\omega} + \sqrt[5]{\omega'} + \sqrt[5]{\omega''} + \sqrt[5]{\omega'''},$$

whose radicand is formed by

$$\omega = \varepsilon_0^2 \varepsilon_0'$$

out of the relative fundamental unit ε_0 of the cyclic biquadratic maximal real subfield $\Omega_0^5 = \mathbb{Q}\left(\sqrt{-e\sqrt{5}\cdot d}\right)$ of $\Omega^5 = \mathbb{Q}\left(\sqrt{-e\sqrt{5}}, \sqrt{d}\right)$, where $e = (1+\sqrt{5})/2$ is the fundamental unit of $\mathbb{Q}_0^5 = \mathbb{Q}(\sqrt{5})$.

Closing remark. Whether in the present case $d = -47$ the disproportionately high fundamental equation can be reduced to a lower one by dividing out of the radicand $\omega = \varepsilon_0^2\varepsilon_0'$ an appropriate 5th power, as was possible in the cases $d = -23$ and $d = -31$ by dividing the radicand ε by $\sqrt{-3}^3$, and whether one can perhaps reduce the equation found here to those coming out of the transformation theory of modular functions (see footnote 1), remains to be seen in a further investigation.

References

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