Teaching and Doing Mathematics with Symbolic Computation

J. F. Ogilvie

Centre for Experimental and Constructive Mathematics,
Department of Mathematics, Simon Fraser University,
8888 University Drive, Burnaby, British Columbia V5A 1S6, Canada
and Escuela de Quimica, Universidad de Costa Rica,
San Pedro de Montes de Oca, San Jose 2060, Costa Rica
electronic mail to 'ogilvie@cecm.sfu.ca'

abstract

The impact of computers on teaching of mathematics is reviewed, to demonstrate a need to reorganise curriculum in mathematics to meet the needs of students of science and engineering in an era of abundant computers and sophisticated software for symbolic computation. Examples are presented of graphical enhancement of teaching of various mathematical concepts.

introduction

Mathematics is the language of science, technology, commerce – of almost every aspect of organised human activity. An acquaintance with mathematics beyond arithmetic is desirable for every adult citizen of any country, but a working knowledge of mathematics is essential for any person involved in a technical career. Although, for some degree courses in science and engineering in the past, study of higher mathematics over several years was compulsory, despite an increasingly theoretical or mathematical content of such courses, in many cases the degree regulations have become relaxed so as to require fewer courses than formerly. How can we, as instructors of mathematics and of other subjects that depend heavily on mathematics, respond to this situation, so as to provide present and future students with a proper and sound foundation for technical mathematical capabilities that they will require in a technical career?

nature of problems in teaching mathematics

As a subject or discipline, mathematics is perceived to be difficult. In recent years I have attended many research lectures or seminars in mathematics, and questions posed to the lecturer afterwards tended to emanate almost exclusively from professors or students with expertise in the same field of mathematics, apart from one or two extraordinary individuals who seem able to proffer intelligent comments or queries across many fields; hence mathematics at the research level is difficult even for professional mathematicians. This situation differs substantially from the situation in research lectures in chemistry and physics, which I have also attended in the same universities over many years, in which questions arise not merely from specialists in the same field. Of course all those professors of mathematics are deemed capable of instructing basic courses constituting a standard undergraduate curriculum. At least at the undergraduate level or below, the difficulty of mathematics lies more in technical details than in fundamental concepts such as a derivative or definite integral, although there are undoubtedly subtleties in even those concepts that a user of mathematics should encounter and appreciate. Especially for students of science and engineering, or economics and other subjects in which mathematical methods play an important role in certain fields, the most important aspect of mathematics is knowledge of what mathematical techniques might be brought to bear on the solution of various problems and of how those techniques might be implemented. If a student understands the mathematical principles and the basis of implementation of a particular approach, including the existence of pathological conditions limiting its effectiveness, details of a particular implementation become subsidiary. For instance, an instructor can emphasize the concept and basis of an indefinite or definite integral; whether a method of partial fractions or substitutions or repeated integration by parts, or some combination of these, serves to evaluate that integral is immaterial when a command to a computer suffices to yield an exact result.

As a simple but non-trivial example, let us consider calculating the square root of a decimal number. When I was a wee lad, my peers and I in Canadian schools were taught by rote how to extract manually such a square root, involving placing the original number in the internal angle of a corner u and subsequently undertaking multiplications and subtractions, introducing two additional digits of the operand at a time, until sufficient precision in the answer is attained. Doubtless many of those present at this conference might still be able to effect this operation, tedious as it is. A few years later we pupils encountered logarithms, involving simple division of a logarithm by two with conversions to obtain that square root with four or five decimal digits, or perhaps two or three digits on a slide rule. Only fifty years after I learned that original procedure did I happen to discover the underlying theory, essentially a binomial expansion, but for half a century I might have blissfully extracted square roots of decimal numbers in ignorance thereof. Somebody mentioned that there is an analogous manual method, but more complicated, for extraction of a cube root. Can anybody here execute that operation? What matters it now? Since 1974 in a major Canadian educational jurisdiction, that manual method to yield even a square root has been eliminated from formal arithmetical curriculum. For both pedagogical and practical purposes, the accepted way to extract a square root of a number is to depress a pertinent button on an ubiquitous and essentially costless pocket calculator. I expect that few persons would doubt my understanding of what is a square root, proper operations on it and its various properties, but all these conditions apply regardless of my capability to undertake manually a particular and monotonous arithmetical procedure. This example involves mere arithmetic. No responsible educator – especially a mathematical educator – can dismiss the value of a child becoming adept at mental arithmetic involving the four standard operations, for which purpose as a child I was exposed to regular drill for several months, or the value of each person being able mentally to estimate roughly either the amount of purchase tax associated with a specified price of a desired article or the net salary after income tax is deducted at a certain rate, or even a square root of a number, but for all practical purposes, when a reasonably precise numerical value is required, a calculator is called into action. Literacy and numeracy remain crucial components of education for life of each and every individual in a modern democratic society; computers must assist such personal development, not replace abilities of independent thinking, in the same way that, despite our individual dependence on private motorcar or public transit to commute between home and place of work or shopping, each of us must maintain personal physical fitness as a requisite of a comfortable and healthy life.

A few decades ago in some countries, for each cashier at a station for payment in a commercial establishment the employer provided an abacus, and not many years ago in other countries even a teller in a chartered bank verified the result from a desk calculator on an abacus. Does an abacus, like a slide rule, exist now outside a museum? During a few decades before 1990, many students were attracted to mathematical studies precisely to avoid computers. Such students have progressed in their careers to become professors who now form likely a majority of academic staff in departments of mathematics in most universities. Even whilst they unhesitatingly employ electronic mail and ‘word processors’ to prepare their personal and professional documents, their antipathy towards computers for mathematical purposes remains unabated, consciously or subconsciously; such reactionary academic staff have much power directly or indirectly to stifle educational innovation, particularly as it would require their own major reassessment and reorganisation of their own methods of teaching standard courses. A few years from now, will such a professor exist outside a museum? Until recently, an instructor of mathematics might be able to employ, when presenting a lecture, notes essentially based on lectures that he or she might have collected as a student a few decades earlier. Mathematics might be timeless, but the relative importance of its fields and the methods of teaching it are no longer invariant   if they ever were. When during a demonstration of some computer lessons to a group of senior professors in 1999 I alluded to teaching mathematics through a succession of theorems, the head of a department of mathematics in a major university retorted swiftly “We do not teach that way any more, even for our own students”, but those same professors were quite capable of resisting consideration of adoption of mathematical software for pedagogical applications. By your presence here, members of this audience demonstrate that you do not belong to at least the rear guard of those who would hinder deployment of instructional technology. After attendance at this meeting, your task and mission is to convey the spirit of our discussion to your own institutions, to inspire your reluctant colleagues to respond to the challenge to prepare students for the future, not the past.

computers and symbolic computation as a solution

Just as arithmetical operations available on a pocket calculator have practically eliminated the need of one’s capacity or capability to do arithmetic manually, mathematical software on an ubiquitous computer has practically eliminated the need of one’s capability to undertake manual mathematical operations up to a senior undergraduate level. A perceptive professor of mathematics in Universidad de Costa Rica recently remarked to me, “For many [mathematician] colleagues it is hard to understand that computers are here forever”. The fact of that ubiquitous computer and also highly sophisticated – even if inevitably imperfect – mathematical software requires that we examine mathematics – not merely its practice but especially its teaching – in a new light. On another occasion that same enlightened professor remarked that, with a computer, he could teach the material of a standard semester course in linear algebra in three weeks. Whilst there might be some hyperbole in the latter assertion, it must not be summarily dismissed. What he and I both appreciate is that an approach to teaching mathematics must be profoundly revised and reorganised on the basis of mathematical technology – hardware and software. During the past half century in which the electronic digital computer has progressed from barely existing to having become even a household commodity, the mathematical needs of students of chemistry, like all other branches of science and engineering, have increased in scope and depth, but the numbers of required courses have significantly diminished, as I comment above. Whereas, when I was a student, analytical chemistry comprised qualitative and quantitative analysis involving only simple chemical compounds and rudimentary glassware, contemporary analytical chemistry has even a new name – chemometrics. Being the chemical equivalent of econometrics for economics, chemometrics is concerned with the control of instruments with computers and the treatment of data generated by and transmitted from those instruments. Computers have major roles in the practice of chemistry in its other branches, just as in all other divisions of science and engineering. Among those several courses over four years that were an obligatory component of my education in chemistry or physics, there was no significant content of statistics, although that field of mathematics naturally underpins chemometrics in its second aspect. One might predict that many approaches to mathematical applications will become modified in the next decade according to an expanding influence of fuzzy logic.

I advocate an holistic view of mathematical needs of students of science and engineering: one must consider the total scope of mathematical fields and applications for any such discipline and design curriculum on that basis. Whereas departments of mathematics in the past offered for students of science and engineering discrete courses of duration one or two semesters on algebra, differential and integral calculus, multivariate and vector calculus, linear algebra, differential equations, statistics et cetera, the fact that practical experience has already demonstrated that one can teach, with symbolic computation, the principles and implementation of material from each such traditional semester course within a lesser duration implies that an overview of the total subject an holistic approach is required. Up to the middle of the past century there were available several textbooks of title such as *College Mathematics*, which purported to cover within several hundred pages all mathematics required by a student of science and engineering. As both mathematicians and instructors of courses in other subjects dependent on mathematics came to recognise, such general courses seemed to fail to yield sufficient proficiency in specific important areas, such as calculus; then general courses, even spread over more than one year, became supplanted by specific courses devoted to calculus, linear algebra, differential equations and so forth, in sequence. When protracted drill necessary for manual proficiency becomes replaced by use of standard commands in software, a much decreased duration of instruction is required to teach a particular field of mathematics. In that way the relations between these mathematical fields can be emphasized better through a common framework of mathematical software than when each field is taught as a separate entity, each with its particular jargon and peculiarities of methods.

You are likely surmising that, through use of symbolic computation, I am promising increased coverage of mathematics in a decreased duration. Yes, you are right, but that objective in no way contravenes the first law of thermodynamics or other universal truth. What a student loses by learning mathematical principles, and subsequent implementation of mathematical operations on a computer, is indisputably the capability of undertaking manually the same operations in any complicated form, just as that button on a pocket calculator has tolled the death knell of extracting a square root by hand. After a traditional course a student might be under a delusion of thinking that he or she understands mathematics when one knows only how to solve exercises. Because a professor of mathematics, or of a subject that incorporates a strongly mathematical component, might have a profound understanding or appreciation of mathematics to a senior undergraduate level, he should not assume an analogous capability of a typical student in his course. A representative student might arguably understand mathematics better when trees of technical details of operations fail to obscure the forest of principles. On a basis of results of standard examinations involving routine problems, a professor of mathematics should not delude himself into imagining that students actually understand profoundly the mathematics involved in those ‘service’ courses for science and engineering – most students do not, just as most professors of mathematics might have only a shallow understanding of research fields of mathematics beyond their field of specialty. What Fred Simons [1], professor at Eindhoven University of Technology in Netherlands, wrote in 1997 is even more relevant and practicable at the end of year 2003: the content of traditional service courses in mathematics has been determined by the needs of users of mathematics in the pre-computer age; service courses in mathematics that concentrate on how to solve exercises can be transformed into courses on how to use software for computer algebra. A service course in mathematics in which computational work is done with a computer must have content and emphasis on skills, and consequently assessment, different from those of a traditional service course. Merely adding computer commands to a traditional course is ineffective: students resent the extra effort required with no perceived benefit at the ultimate examination. What a student loses in manual capability he or she gains by being able to solve significant problems, not merely simplified or prototypical cases suitable for manual solution within a limited period – two or three hours – of a conventional written examination. If a course involves performance of mathematical operations with a computer, assessment of that course must be likewise based on performance with that computer. An instructor of such a course based on symbolic computation must reorganise fully his or her point of view: lectures must include demonstration of implementation of mathematical principles with projection from a computer, and practical periods supervised by persons knowledgeable in both mathematics and software become an intrinsic component of such a course. Instructors of courses in other subjects that depend on such service courses in mathematics must be prepared to alter their outlook on mathematical capabilities of students, accepting or requiring use of computers in their own exercises and examinations, whilst concurrently the problems treated expand from trivial cases dictated by manual mathematical limitations to realistic contexts.

Teaching mathematics with software for symbolic computation allows an instructor to explore a topic or principle from four points of view: a formal statement in words, just as according to tradition, including emphasis on definitions of terms; an algebraic or symbolic treatment, which can expand to take advantage of the speed and scope of software for algebraic operations; numerical aspects, with test cases over a large range, with numerical examples used to introduce topics as much as practicable; graphics, showing geometrical interpretations in two or three dimensions, with animations, in a way that is entirely impracticable with traditional chalk on a blackboard. The advantage of the latter can not be overestimated: a picture is certainly worth a thousand words of jargon, and makes a concept memorable to even a mathematically disinclined student. The capacity of contemporary software for symbolic computation to produce outstanding plots is astounding; teaching of mathematics without use of such displays is unacceptably inferior. In a mathematics course, emphasis on concepts and reasoning can replace drill on technical details of manipulation required to solve routine exercises, and plots of geometric constructions can underpin those concepts and critically enliven the reasoning.

examples of use of symbolic computation in teaching mathematics

I illustrate a few aspects of teaching mathematics in various areas, employing Maple software [2] for this purpose. Maple was developed originally at University of Waterloo in Canada primarily to assist students of science and engineering to undertake mathematical operations on a computer in a way that a Fortran compiler or Basic interpreter enables execution directly of merely arithmetical operations; although it has become a major commercial product, its devotion to an educational mission remains steadfast, and at present Maple sets a standard according to which other mathematical software can be assessed. Freely available software that is readily acquired through internet includes comprehensive courses in traditional areas of mathematics, such as algebra, calculus, linear algebra, ordinary and partial differential equations, apart from applications in many areas of science and engineering.

In calculus the Riemann sum underpins the concept of a definite integral; evaluation of an area under a curve as a sum of areas of rectangles as the width of each rectangle decreases lends itself well to animation. In multivariate calculus the directional properties of a partial derivative acquire a profound meaning when one can view curves on a surface in three dimensions. Many operations in linear algebra are conceptually simple but exceedingly tedious if restricted to manual execution; we practise gaussian elimination on a matrix of significant order directly in real time to understand the solution of linear equations, and we can even plot a matrix. Graphic depiction of eigenvectors facilitates comprehension of the nature of eigenvalues. When one plots a direction field for a differential equation of first order, its meaning becomes clear in a way that mere algebraic presentation never achieves. Statistical concepts and practices are facilitated with symbolic software, aided by graphical depiction, in a way that mere numerical software can not approach; a sequential simplex method of optimisation is illustrated on a surface of χ^2 to demonstrate approach to a convergent solution of fitted data.

There is a spectrum of possible approaches to the teaching of mathematics with symbolic computation; apart from no deployment at all, one might merely use software interactively in an occasional lecture demonstration availing of the graphical capability, whereas the other limit of teaching principles and using software for practically all implementation during a course, its assigned exercises and final examination is likely the most effective use of the computer and prepares best the student for future applications. Although generally courses in mathematics at university level begin typically with calculus, when teaching with software for computer algebra one must precede new mathematics with basic numerical and algebraic operations, including solving equations, elementary functions – polynomial, rational, exponential, logarithmic, various trigonometric – and plotting as a mechanism to have a student become familiar with both the notion of symbolic computation and actual rudimentary commands required for routine handling of numbers or symbolic expressions. On that basis one proceeds to teach all standard topics in differential and integral calculus involving a single variable – including numerical techniques commonly neglected in a traditional course, multivariate calculus, linear algebra from matrix and vector through eigenvalue to tensor and vector calculus, ordinary and partial differential – and integral – equations, and statistical topics including regression and optimisation. Brief exposures to other aspects of technical applications of computers pertinent to those students, such as spreadsheets and text editors – whether or not included within the capabilities of the software for symbolic computation, is also practicable within the same stream of courses. A realistic duration of such a programme of instruction would involve at least 80 hours of formal instruction, optimally 120 hours, or a greater period depending on capabilities of students and their previous experience with mathematics and computers; those hours of instruction would normally comprise two hours per week of formal lecture and demonstration and two hours of supervised practice in a computer laboratory, thus running over two or three semesters. One should assume, and arrange accordingly, that students have unsupervised access to the same software for symbolic computation in computer laboratories for additional practice and applications as required, and that students would likely also have software for computer algebra – not necessarily the same product as that in an institutional setting – installed on their personal computers or elsewhere.

summary

We must all agree not only that computers are here forever but also that they affect strongly the teaching and practice of mathematics, just like every other aspect of knowledge activity and communication; hence our student who fails to become significantly acquainted with mathematical – not merely arithmetical – software is not being properly prepared for a technical career. Currently available mathematical software, for symbolic computation with associated numerical and graphical capabilities highly developed, provides a valuable tool for both teaching and doing mathematics, and should become an integral component of routine instructional presentation. Instruction should emphasise mathematical principles, with numerical and graphical interpretations and illustrations, and indicate how mathematical operations are implemented, although there is no necessity to restrict implementation to a single software product. Students should learn how to adopt an experimental and constructive approach to mathematics, based on mathematical software, rather than a sterile description according to theorems, corollaries, lemmas et cetera. As mathematical software continues to evolve, both students and their instructors must expect to expand their mathematical horizons and to progress in their own development stimulated through that software. The future development of internet communication and its impact on education is difficult to predict – even a few years into the future, but what is certain is that both content and process of education, including mathematical education, is evolving rapidly as a consequence of the existence and deployment of digital computers and symbolic computation. Each instructor of mathematics has a solemn duty and responsibility to adapt to, and to work with, computers to prepare optimally his or her students for future technical careers and even life styles. The future might be unpredictable in detail but the trends are clear: computers and symbolic computation in teaching and practice of mathematics are indisputably part of them.

references

[1] F. Simons, Computer Algebra in Service Courses, CTI Math & Stats Newsletter, 8 (3), 1997 August, CTI Centres for Mathematics and Statistics, University of Birmingham, UK http://www.bham.ac.uk/ctimath/newsletter/vol8.htm on internet

[2] http://www.maplesoft.com/ and http://www.mapleapps.com on internet