GENERAL FORMS FOR MINIMAL SPECTRAL VALUES FOR A CLASS OF QUADRATIC PISOT NUMBERS.

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ABSTRACT. We study the spectrum of real numbers that result when all height one polynomials are evaluated at a Pisot number. This continues the research theme initiated by Erdős, Joó and Komornik [6]. We are particularly interested in the minimal non-zero value of this spectrum. Formally we denote this value as $l^1(q)$, and extend this definition to all height m polynomials as

$$l^m(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n, \epsilon_i \in \mathbb{Z}, |\epsilon_i| \leq m, y \neq 0).$$

A recent result of Komornik, Loreti and Pedicini [13] gives a complete description of $l^m(q)$ when q is the Golden ratio. This paper extends this result to include all unit quadratic Pisot numbers. A main theorem is

Theorem. *Let q be a quadratic Pisot number that satisfies a polynomial of the form $p(x) = x^2 - ax \pm 1$, with conjugate r. If q has a continued fraction approximations $\left\{\frac{C_k}{D_k}\right\}$ and k is the maximal integer such that*

$$|D_k r - C_k| \leq m \frac{1}{1 - |r|}$$

then

$$l^m(q) = |D_k q - C_k|.$$

A value related to $l(q)$ is $a(q)$, the minimal non-zero value when all ± 1 polynomials are evaluated at q [14]. Formally this is

$$a(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^2 + \cdots + \epsilon_n q^n, \epsilon_i = \pm 1, y \neq 0).$$

An open question of [2] concerning how often $a(q) = l(q)$ is also answered here.

1. INTRODUCTION

Erdős, Joó and Komornik in 1990 [5] initiate the study of the spectra resulting from evaluating certain classes of polynomials at values $q > 1$. It is known that

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if we evaluate a height m polynomial at a Pisot number, then it is either zero, or bounded away from zero [3, 8, 11]. Thus we have that the infimums of these spectra being studied are greater than zero for all Pisot numbers. To this end Erdős, Joó and Joó [7] define these infimums as $l^m(q)$.

Definition 1. Define $l^m(q)$ as:

$$l^m(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n, \epsilon_i \in \mathbb{Z}, |\epsilon_i| \leq m, y \neq 0).$$

A related area of interest is the case where the class of polynomials is restricted to polynomials with ± 1 coefficients [2, 14]. The minimal value in this case is defined as $a(q)$.

Definition 2. Define $a(q)$ as:

$$a(q) := \inf(|y| : y = \epsilon_0 + \epsilon_1 q^1 + \cdots + \epsilon_n q^n, \epsilon_i = \pm 1, y \neq 0).$$

An open question concerning when $a(q) = l(q)$ is answered in Section 5. For additional history of problems relating to $l^m(q)$ and $a(q)$, see [2, 12].

Specific values of $l^m(q)$ are calculated for some Pisot numbers q. If q is the Pisot number that satisfies $q^3 - q^2 - 1$, then $l(q) = q^2 - 2$ [13]. If q is the Pisot number satisfying $q^n - q^{n-1} - \cdots - 1$ then $l(q) = \frac{1}{q}$ [7, 13]. If q is the Golden ratio, (the greater root of $x^2 - x - 1$) then $l^2(q) = q^3 - 2q^2 + 2q - 2 = 3 - 2q$ (this corrects a misprint in [3], which used the notation $\liminf(u_n^{(2)})$ for $l^2(q)$). For general m, and q the Golden ratio, all $l^m(q)$ are known. If F_k is the kth Fibonacci number ($F_0 = 0, F_1^r = 1, F_n = F_{n-1} + F_{n-2}$), and $q^{k-2} < m \leq q^{k-1}$ then $l^m(q) = |F_k q - F_{k+1}|$ [13].

In [2] an algorithm is given to calculate $l^m(q)$ for any Pisot number q and any integer m, limited only by the memory of the computer. Although this method can make calculations for any given q and m, it will only solve the problem for specific examples. A tabulation of other $l^m(q)$ for various m and q, based on these methods of calculation, is found at [10]. Upon examination of these tables, we find another pattern, similar to that of $l^m(q)$ when q is the Golden ratio. This pattern concerns $l^m(q)$ for unit quadratic Pisot numbers. Recall:

Definition 3. A *Pisot number* is a real algebraic integer, all of whose conjugates are of modulus strictly less than 1. A *Pisot polynomial* is the minimal polynomial of a Pisot number.

The class of Pisot numbers we consider is:

Definition 4. Let $\mathcal{P}$ be the set of unit quadratic Pisot numbers. This is easily seen to be the appropriate roots of polynomials of the form $x^2 - rx - 1$ for $r = 1, 2, 3, \cdots$, and $x^2 - rx + 1$ for $r = 3, 4, 5, \cdots$.

Let $q \in \mathcal{P}$. This paper shows that $l^m(q) = |Dq - C|$ where $\frac{C}{D}$ is a convergent of the continued fraction of q. A better description of which convergent $l^m(q)$ is equal to is given in Theorem 2.1, in Section 2.

2. A description of $l^m(q)$

In this section we give a description of $l^m(q)$ for all $q \in \mathcal{P}$. First though, we need a few lemmas and definitions.

Definition 5. We define the sequences $\{A_n^{a,b}\}_{n=0}^{\infty}$ and $\{B_n^{a,b}\}_{n=0}^{\infty}$ as

1. $A_0^{a,b} = 0,\ A_1^{a,b} = 1,\ A_n^{a,b} = aA_{n-1}^{a,b} + bA_{n-2}^{a,b}$,
2. $B_0^{a,b} = 1,\ B_1^{a,b} = 0,\ B_n^{a,b} = aB_{n-1}^{a,b} + bB_{n-2}^{a,b}$.

When the a and b can be inferred from context, we use the notation A_n and B_n.

Lemma 1. *Using the notation of Definition 5*

$$\det\left(\begin{bmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{bmatrix}\right) = (-b)^{n-1}$$

Proof:

$$\begin{aligned}
\det\left(\begin{bmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{bmatrix}\right) &= \det\left(\begin{bmatrix} aA_{n-1} + bA_{n-2} & A_{n-1} \\ aB_{n-1} + bB_{n-2} & B_{n-1} \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} bA_{n-2} & A_{n-1} \\ bB_{n-2} & B_{n-1} \end{bmatrix}\right) \\
&= -b\det\left(\begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix}\right) \\
&\ \vdots \\
&= (-b)^{n-1}\det\left(\begin{bmatrix} A_1 & A_0 \\ B_1 & B_0 \end{bmatrix}\right) \\
&= (-b)^{n-1}\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= (-b)^{n-1}
\end{aligned}$$

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Lemma 1 can be extended fairly easily to higher dimensions.

Lemma 2. *For $n \geq 0$ we have*

$$x^n \equiv A_n^{a,b} x + B_n^{a,b} \pmod{x^2 - ax - b}.$$

Proof: Simple induction argument.

$\blacksquare$

Lemma 3. *Let q and r be the two roots of $x^2 - ax - b$. Then*

$$A_n^{a,b} = \frac{1}{q-r} q^n + \frac{1}{r-q} r^n$$

and

$$B_n^{a,b} = \frac{r}{r-q} q^n + \frac{q}{q-r} r^n.$$

Proof: This is a standard result from recurrence relations, see for example [9].

$\blacksquare$

By combining Lemmas 2 and 3 we get:

Lemma 4. *Let q and r be the two roots of $x^2 - ax - b$. Then*

$$x^n \equiv \frac{1}{q-r} q^n (x - r) + \frac{1}{r-q} r^n (x - q) \pmod{x^2 - ax - b}.$$

The next result is well known in the literature on continued fractions, see for example [4].

Lemma 5. *Let q be a real number, and m some integer, then the best approximation to q by $\frac{C}{D}$, where $0 < C, D \leq m$ is a convergent $\frac{C_n}{D_n}$ of the continued fraction of q, for n maximal such that $C_n, D_n \leq m$.*

The next lemma is reminiscent to those lemmas in Section 3 of [13], but the presentations is different. For this lemma we need the following definition:

Definition 6. Define $\widehat{\mathbb{R}}[x] = \{p \in \mathbb{R}[x] : H(p) \leq 1\}$.

Lemma 6. *Let $p(x) = x^2 - ax \pm 1$ be associated with some $q \in \mathcal{P}$. Let m be such that $|A_n| \leq m|B_n|$ for all $n \geq 2$. Let $y' \in m\widehat{\mathbb{R}}[x]$ such that $y' \equiv cx + b \pmod{p(x)}$ where $c, b \in \mathbb{Z}$. Then there exists a $y \in m\widehat{\mathbb{R}}[x] \cap \mathbb{Z}[x]$ such that $y \equiv cx + b \pmod{p(x)}$.*

Proof: First it is worth noting that we can always find an m such that $|A_n| \leq m|B_n|$ as $\lim_{n\to\infty} \left|\frac{A_n}{B_n}\right| = \left|\frac{1}{r}\right| > 1$.

By assuming that $y \equiv cx + b \pmod{p(x)}$ with $c, b \in \mathbb{Z}$, we show here how to find y' such that $y' \equiv y \pmod{p(x)}$ and $y' \in \mathbb{Z}[x] \cap m\widehat{\mathbb{R}}[x]$. Let

$$y = a_n x^n + \cdots a_0.$$

If $a_n \in \mathbb{Z}$ then continue inductively on $a_{n-1}x^{n-1} + \cdots + a_0$. If $a_0 \in \mathbb{Z}$ then continue inductively on $a_n x^{n-1} + \cdots + a_1$. If neither a_0 nor a_n is an integer, then use the identity that $x^n - A_n x - B_n \equiv 0 \pmod{p(x)}$ along with the fact that $|A_n| \leq m|B_n|$ to solve for α where $a_n + \alpha \in \mathbb{Z}$ or $a_0 - \alpha B_n \in \mathbb{Z}$, and $|a_n + \alpha|, |a_1 - A_n\alpha|, |a_0 - B_n\alpha| \leq m$. Continue inductively on $a_n x^n + \cdots + a_0 + \alpha(x^n - A_n x - B_n)$.

By repeated application of this we see that y' is such that all of the a_i are integers with the possible exception of two consecutive terms, a_j and a_{j-1}. Notice that $a_n x^n + \cdots + a_{j+1}x^{j+1} + a_{j-2}x^{j-2} + \cdots a_0 \equiv c'x + b'$ for some $b', c' \in \mathbb{Z}$. Thus we see that $a_j x^j + a_{j-1}x^{j-1} \equiv (c - c')x + (b - b')$, where $c - c', b - b' \in \mathbb{Z}$. ¿From Lemma 2 we know that $a_j x^j + a_{j-1}x^{j-1} \equiv a_j A_j x + a_{j-1}A_{j-1}x + a_j B_j + a_{j-1}B_{j-1}$. Thus we have that

$$\begin{bmatrix} A_j & A_{j-1} \\ B_j & B_{j-1} \end{bmatrix} \begin{bmatrix} a_j \\ a_{j-1} \end{bmatrix} = \begin{bmatrix} c - c' \\ d - d' \end{bmatrix}$$

By noticing that the determinate of $\begin{bmatrix} A_j & A_{j-1} \\ B_j & B_{j-1} \end{bmatrix}$ is ± 1, we get that the inverse of this matrix is integral, and thus $a_j, a_{j-1} \in \mathbb{Z}$.

■

What is interesting here is that this proof is constructive, and a computer algorithm can be designed from this. This is described in Section 3.

Theorem 2.1. *Let $q \in \mathcal{P}$ satisfy a polynomial of the form $p(x) = x^2 - ax \pm 1$, with conjugate r. If q has a continued fraction approximations $\left\{\frac{C_k}{D_k}\right\}$ and k is the maximal integer such that*

$$|D_k r - C_k| \leq m \frac{1}{1 - |r|}$$

then

$$l^m(q) = |D_k q - C_k|.$$

It is worth noting, when q is the Golden ratio then the result is equivalent to Theorem 3.1 in [13].

Proof: Recall from Lemma 4 that

$$x^n \equiv \frac{1}{q-r} q^n (x-r) + \frac{1}{r-q} r^n (x-q).$$

From this is follows that

$$\widehat{\mathbb{R}}[x] \equiv v(x-r) + w(x-q) \pmod{p(x)}$$

where $v \in \mathbb{R}$ and $|w| \leq \frac{1}{(q-r)(1-|r|)}$.

Consider the continued fraction of q, $\left\{\frac{C_k}{D_k}\right\}$. Lemma 5 indicates that the best linear terms are of the form $D_k q - C_k$. Lemma 6 indicates that if $D_k x - C_k \in m\widehat{\mathbb{R}}[x]$ then there exists a $y \in \mathbb{Z}[x] \cap \widehat{\mathbb{R}}[x] \pmod{p(x)}$ such that $y \equiv D_k x - C_k \pmod{p(x)}$. It follows that $l^m(q) = D_k q - C_k$ when $D_k x - C_k \in m\widehat{\mathbb{R}}[x] \pmod{p(x)}$ with k maximal. Write $D_k x - C_k$ as $v(x-r) + w(x-q)$. As $D_k x - C_k \in m\widehat{\mathbb{R}}[x] \pmod{p(x)}$ we have $w \leq \frac{m}{(q-r)(1-|r|)}$. Thus

$$\begin{aligned} |D_k r - C_k| &= |v(r-r) + w(r-q)| \\ &= |w(q-r)| \\ &\leq \frac{m}{(q-r)(1-|r|)}(q-r) \\ &\leq \frac{m}{1-|r|} \end{aligned}$$

This is the desired result.

■

Corollary 1. *Define $F_n = rF_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$ and q the Pisot root of $x^2 - rx - 1$. If $q^{k-1}(q-1) \leq m < q^k(q-1)$ then $l^m(q) = |F_k q - F_{k+1}|$.*

Proof: It is easy to verify that $\left\{\frac{F_{k+1}}{F_k}\right\}$ are the continued fractions of q. A simple calculation shows that

$$F_k = \frac{1}{q-r}(q^k + r^k)$$

and that $r = \frac{-1}{q}$. This yields that $l^m(q) = |F_k q - F_{k+1}|$ when

$$\begin{aligned}
|F_k r - F_{k+1}| &\leq \frac{m}{1-|r|}, \\
\left| \frac{1}{q-r}((q^k + r^k)r - (q^{k+1} - r^{k+1})) \right| &\leq \frac{m}{1-|r|}, \\
\left| \frac{1}{q-r}(q^k r - q^{k+1}) \right| &\leq \frac{m}{1-|r|}, \\
\left| \frac{r-q}{q-r} q^k \right| &\leq \frac{m}{1-|r|}, \\
q^k &\leq \frac{m}{1-|r|}, \\
(q-1)q^{k-1} &\leq m,
\end{aligned}$$

and the result follows.

■

Table 1 gives the ranges that m is in, for $l^m(q) = F_{k-1}q - F_k$.

$l^m(q)$	q^2-q-1	q^2-2q-1	q^2-3q-1	q^2-4q-1	q^2-5q-1
$\lvert F_0 q - F_1\rvert$		[1,1]	[1,2]	[1,3]	[1,4]
$\lvert F_1 q - F_2\rvert$	[1,1]	[2,3]	[3,7]	[4,13]	[5,21]
$\lvert F_2 q - F_3\rvert$		[4,8]	[8,25]	[14,58]	[22,113]
$\lvert F_3 q - F_4\rvert$	[2,2]	[9,19]	[26,82]	[59,245]	[114,586]
$\lvert F_4 q - F_5\rvert$	[3,4]	[20,48]	[83,274]	[246,1042]	[587,3048]
$\lvert F_5 q - F_6\rvert$	[5,6]	[49,115]	[275,904]	[1043,4413]	[3049,15826]
$\lvert F_6 q - F_7\rvert$	[7,11]	[116,280]	[905,2989]	[4414,18698]	[15827,82183]
$\lvert F_7 q - F_8\rvert$	[12,17]	[281,675]	[2990,9871]	[18699,79205]	[82184,426742]
$\lvert F_8 q - F_9\rvert$	[18,29]	[676,1632]	[9872,32605]	[79206,335521]	[426743,2215893]

Table 1: Relation between $l^m(q)$ and $F_{k-1}q - F_k$.

With a proof similar to that of Corollary 1, we get

Corollary 2. *Define $E_n = rE_{n-1} - E_{n-2}$ with $E_0 = 0$ and $E_1 = 1$ and $G_n = rG_{n-1} - G_{n-2}$ with $G_0 = 1$ and $G_1 = 1$ and and q the Pisot root of $x^2 - rx + 1$. If $q^{k-3}(q-1)^2 \leq m < q^{k-2}(q-1)$ then $l^m(q) = |G_{k-1}q - G_k|$ and if $q^{k-2}(q-1) \leq m < q^{k-2}(q-1)^2$ then $l^m(q) = |E_{k-1}q - E_k|$.*

Table 2 gives the ranges that m is in, for $l^m(q) = |G_{k-1}q - G_k|$ and $l^m(q) = |E_{k-1}q - E_k|$

$l^m(q)$	q^2-3q+1	q^2-4q+1	q^2-5q+1	q^2-6q+1	q^2-7q+1
$\lvert G_0q-G_1\rvert$					
$\lvert E_0q-E_1\rvert$	[1,1]	[1,2]	[1,2]	[1,3]	[1,5]
$\lvert G_1q-G_2\rvert$			[3,3]	[4,4]	
$\lvert E_1q-E_2\rvert$	[2,2]	[3,7]	[4,14]	[5,23]	[6,34]
$\lvert G_2q-G_3\rvert$	[3,4]	[8,10]	[15,18]	[24,28]	[35,40]
$\lvert E_2q-E_3\rvert$	[5,6]	[11,27]	[19,68]	[29,135]	[41,234]
$\lvert G_3q-G_4\rvert$	[7,11]	[28,38]	[69,87]	[136,164]	[235,275]
$\lvert E_3q-E_4\rvert$	[12,17]	[39,103]	[88,329]	[165,791]	[276,1609]

Table 2: Relation between $l^m(q)$, $E_{k-1}q - E_k$ and $G_{k-1}q - G_k$.

3. Finding the height m polynomials

For $q \in \mathcal{P}$ with minimal polynomial $p(x)$, we have that $l^m(q) = |Dq - C|$ for some integers C and D where $\frac{C}{D}$ is a convergent of q. What this section is interested in is finding the particular height m polynomial that $l^m(q)$ relates to. We notice that Lemma 6 can be implemented into an algorithm. Thus it is sufficient to find a $t(x) \in m\widehat{\mathbb{R}}[x]$ such that $t(x) \equiv Dx - C \pmod{p(x)}$. For this we can use the simplex method [15]. Write

$$Dx - C + (\sum_{k=0}^{n} a_k x^k)p(x) = \sum_{k}^{n+2} b_k x^k$$

for unknowns a_k. We wish $-m \le b_k \le m$ for all $k = 0, \cdots, (n+2)$. So for the correct value of n we minimize for h with

$$-h \le b_k \le h \tag{1}$$

and solve for the a_k. A careful calculation can yield the minimal value for n that works as

$$n = \left\lfloor \ln\left(\frac{m + |Dr - C|\,|r| - |Dr - C|}{m\,|r|}\right)(\ln(|r|))^{-1}\right\rfloor \tag{2}$$

Using the simplex method in this way works for any polynomial, where as the value for n given in equation (2) is only true for $q \in \mathcal{P}$. Thus if we wish to implement this algorithm for polynomials that come from some $q \notin \mathcal{P}$ we can simply take n increasing until we find one that works.

We now do an example

Example 1. Let q be the root of $q^2 - 3q + 1$. A simple calculation demonstrates that $l^7(q) = 5q - 13$ Using equation (2) we have that the minimal value for n is

3. So minimizing h with respect to the constraints in equation (1) as $n = 3$ gives $h = \frac{305}{44} < 7$. This gives a polynomial of:

$$\frac{17}{4}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{305}{44}x^2 - \frac{305}{44}x - \frac{305}{44}$$

Here we use the techniques in Lemma 6 iteratively. Notice that at any step, only three coefficients are altered.

$$\begin{aligned}
\frac{17}{4}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{305}{44}x^2 - \frac{305}{44}x - \frac{305}{44} &\equiv 5x - 13 \pmod{x^2 - 3x + 1},\\
\frac{327}{77}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{305}{44}x^2 - \frac{520}{77}x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1},\\
\frac{371}{88}x^5 - \frac{305}{44}x^4 - \frac{305}{44}x^3 - \frac{553}{88}x^2 - 7x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1},\\
4x^5 - \frac{305}{44}x^4 - \frac{229}{44}x^3 - \frac{305}{44}x^2 - 7x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1},\\
4x^5 - 7x^4 - 5x^3 - 7x^2 - 7x - 7 &\equiv 5x - 13 \pmod{x^2 - 3x + 1}.
\end{aligned}$$

Thus we have found a height 7 integer polynomial $p(x)$ where $l^7(q) = 5q - 13 = p(q)$.

4. Non-unit quadratic Pisot numbers

It is worth noting that Theorem 2.1 does not work for all quadratic Pisot numbers. The problem is that Lemma 6 doesn't work for all Pisot polynomials $x^2 - ax - b$. For example if q is the Pisot root of $x^2 - 2x - 2$ (of approximately 2.732050808) then we see that

$$\frac{7}{16}x^8 - 2x^7 + 3x^6 - 3x^5 + 3x^4 - 3x^3 + 3x^2 - 3x + 3 \equiv 8 - 3x \pmod{x^2 - 2x - 2}$$

is in $3\widehat{\mathbb{R}}[x]$ but it is not in $\mathbb{Z}[x] \cap 3\widehat{\mathbb{R}}[x]$. Worse we have that $|8 - 3q| = 0.196152424 < 0.267949192 = l^3(q)$.

5. The existence of an infinite family of Pisot numbers where $l(q) = a(q)$

It is easy to give an infinite set of Pisot numbers q where $l(q) = a(q)$. We know from [7] that if $q^n - q^{n-1} - \cdots - 1 = 0$ then $l(q) = q^{n-1} - q^{n-2} - \cdots - 1$. It is clear in this case that $l(q) = a(q)$. This answers question 1 in [2] in the negative, as it gives an infinite family of Pisot numbers where $l(q) = a(q)$.

6. Further research

The counter example of Section 4 shows that Theorem 2.1 does not work for all quadratic Pisot numbers. Despite this, the spirit of Theorem 2.1 appears to be true. Computationally $l^m(q) = |Dq - C|$ for $\frac{C}{D}$ a convergent of the continued fraction of q (just not always the one that would be predicted by Theorem 2.1). It would be interesting to know if this is indeed the case.

Secondly it would be of interest if a Lemma similar to Lemma 6 could be found that would work for all polynomials p where $p(0) = \pm 1$, regardless of the degree of $p(x)$. If something like this could be found then this could be used to prove, for $q \in (1, 2)$, that $l(q) > 0$ if and only if q is Pisot. This is conjectured to be true by a number of people, see for example [2, 12]. The second part of this Lemma easily extends to arbitrarily degree, but it is not clear that there is an algorithm that forces all but d consecutive terms to be integers (where d is the degree of $p(x)$).

While searching for patterns among various Pisot numbers, it appears that a nice description exists for the Pisot roots of $x^3 - x - 1$ and $x^3 - x^2 - 1$. Some work is done on this [1] but this is not fully answered as of yet.

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