A new edge selection heuristic for computing the Tutte polynomial of an undirected graph.

Michael Monagan[1†]

[1] *Department of Mathematics, Simon Fraser University, Burnaby, B.C., V5A 1S6, CANADA.*

Abstract. We present a new edge selection heuristic and vertex ordering heuristic that together enable one to compute the Tutte polynomial of much larger sparse graphs than was previously doable. As a specific example, we are able to compute the Tutte polynomial of the truncated icosahedron graph using our Maple implementation in under 4 minutes on a single CPU. This compares with a recent result of Haggard, Pearce and Royle whose special purpose C++ software took one week on 150 computers.

Résumé. Nous présentons deux nouvelles heuristiques pour le calcul du polynôme de Tutte de graphes de faible densité, basées sur les principes de sélection d'arêtes et d'arrangement linéaire de sommets, et qui permettent de traiter des graphes de bien plus grande tailles que les méthodes existantes. Par exemple, en utilisant notre implémentation en Maple, nous pouvons calculer le polynôme de Tutte de l'isocahédron tronqué en moins de 4 minutes sur un ordinateur à processeur unique, alors qu'un programme ad-hoc récent de Haggard, Pearce et Royle, utilisant 150 ordinateurs, a nécessité une semaine de calcul pour obtenir le même résultat.

Keywords: Tutte polynomials, edge deletion and contraction algorithms, NP-hard problems.

1 Introduction

Let G be an undirected graph. The Tutte polynomial of G is a bivariate polynomial $T(G,x,y)$ which contains information about how G is connected. We recall Tutte's original definition for $T(G,x,y)$. Let $e=(u,v)$ be an edge in G. Let $G-e$ denote the graph obtained by deleting e and let $G\,/\,e$ denote the graph obtained by contracting e, that is, first deleting e then joining vertexes u and v. Figure 1 shows an example of edge deletion and contraction.

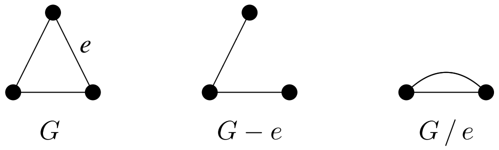

Fig. 1: Graph edge deletion and contraction.

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Definition 1 *Let G be a connected undirected graph. The* Tutte polynomial $T(G,x,y)$ *is the bivariate polynomial defined by*

$$T(G) = \begin{cases} 1 & \textit{if } |E| = 0, \\ x\,T(G/e) & \textit{if } e \textit{ is a cut-edge in } G, \\ y\,T(G-e) & \textit{if } e \textit{ is a loop in } G \\ T(G-e) + T(G/e) & \textit{if the edge } e \textit{ is neither a loop nor a cut-edge in } G. \end{cases} \tag{1}$$

This definition immediately gives a recursive algorithm for computing $T(G,x,y)$. In general, a naive implementation of the algorithm will make an exponential number of recursive calls because of the last case in (1). If G has n vertexes and m edges, the number of recursive calls $C(n+m)$ is bounded by

$$C(n+m) \leq C(n+m-1) + C(n-1+m-1).$$

This is the Fibonacci recurrence. Hence $C(n+m) \in O(1.618^{n+m})$. If G is not biconnected one can apply the following theorem to reduce $C(n+m)$.

Theorem 1 (Tutte [10]) *Let G be a graph with m biconnected components (blocks) $B_1, B_2, \ldots, B_m$. Then $T(G,x,y) = \Pi_{i=1}^{m} T(B_i,x,y)$.*

Another way to reduce $C(n+m)$ is to "remember" the Tutte polynomials computed in the computation tree and use a graph isomorphism test to test whether a graph in the computation tree has been seen before. In [6], Haggard, Pearce and Royle present timings for random cubic and quartic graphs, complete graphs, and random graphs with varying edge densities $0 < p < 1$ that shows that employing graph isomorphism is very effective. For example, it roughly increases by 50% the size of random cubic graphs that can be handled in a given time. A factor determining the effectiveness of the isomorphism test is the order in which the edges are selected. In [9], Haggard, Pearce and Royle investigate various edge ordering heuristics. Two heuristics, which they call MINDEG and VORDER, are found to perform consistently better than random selection.

Our paper is organized as follows. In section 2 we describe the MINDEG and VORDER heuristics and present a new edge selection heuristic. The VORDER heuristic, and our new heuristic, also depend on the ordering of the vertexes in G. We present an ordering that we have found works particularly well with our edge selection heuristic. In section 3 we describe our Maple implementation and explain how we test for isomorphic graphs in the computation tree. In section 4 we present benchmarks comparing the three heuristics with and without the new vertex ordering and with and without an explicit graph isomorphism test. The data presented shows that our new heuristic again, roughly increases by 50% the size of sparse cubic graphs that can be handled in a given time. An experimental finding in this paper is that our new edge selection heuristic, when combined with our vertex ordering, does not require an explicit isomorphism test; a simple test for identical graphs is sufficient.

We end the introduction with some further information about available software for computing Tutte polynomials and related polynomials. Useful references include the very good Wikipedia webpage http://en.wikipedia.org/wiki/Tutte_polynomial and Bollobás' text [1]. The graph theory packages in Mathematica and Maple include commands for computing Tutte polynomials. The Mathematica algorithm does not look for identical or isomorphic graphs in the computation tree (see [6]). The TuttePolynomial command in Maple 11 and more recent versions (see [4]) uses the VORDER heuristic and hashing to test for identical graphs in the computation tree. The fastest available software for computing Tutte polynomials and related polynomials is that of Haggard, Pearce and Royle [9, 6]. It is available on David

Pearce's website at `http://homepages.ecs.vuw.ac.nz/~djp/tutte/`. It uses the canonical graph ordering available in Brendan Mckay's nauty package (see [7]) to identify isomorphic graphs.

We recall the definition for the reliability polynomial and chromatic polynomial.

Definition 2 *Let G be an undirected graph. The* reliability polynomial *of G, denoted $R_p(G)$, is the probability that G remains connected when each edge in G fails with probability p. The* chromatic polynomial *of G, denoted $P_\lambda(G)$, counts the number of ways the vertexes of G can be colored with λ colors.*

For example, $R_p($ •—• $) = 1 - p$ and $P_\lambda($ •—• $) = \lambda(\lambda - 1)$. The reliability and chromatic polynomials can also be computed by the edge deletion and contraction algorithm (see [5]). If G has n vertexes and m edges, they are related to the Tutte polynomial as follows:

$$R_p(G) = (1-p)^{(n-1)}\, p^{(m-n+1)}\, T(G, 1, p^{-1}), \tag{2}$$

$$P_\lambda(G) = (-1)^{(n-1)}\, \lambda\, T(G, 1-\lambda, 0). \tag{3}$$

Since graph coloring is NP−complete, it follows that computing the the chromatic polynomial is NP−hard. Thus (3) implies computing the Tutte polynomial is also NP−hard. It is also known that computing $R_p(G)$ is NP-hard (see [8]). This does not mean, however, that computing the Tutte polynomial for a given graph is not polynomial time. Our new edge selection heuristic is polynomial time for some structured sparse graphs.

2 Edge selection heuristics.

In applying the identity $T(G) = T(G-e) + T(G/e)$ we are free to choose any edge which is neither a cut-edge nor a loop. [If G has a cut-edge or loop, then those edges should be processed first.] In [9], Haggard, Pearce and Royle propose two heuristics, the minimum degree heuristic (MINDEG) and the vertex order heuristic (VORDER). We describe the heuristics here and introduce our new heuristic which is a variation on VORDER.

2.1 The minimum degree heuristic: MINDEG

Consider the graph G in Figure 2.

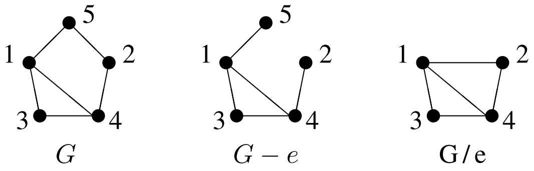


Fig. 2: The minimum degree heuristic.

The minimum degree heuristic picks the edge $e = (u, v)$ where u is the first vertex of minimum degree ($u = 2$ in the example) and v is the first vertex adjacent to u of minimum degree ($v = 5$ in the example). Hence $e = (2, 5)$ is chosen. Shown in the figure are the graphs $G - e$ and $G\,/\,e$. The reader can see that the next edge that will be selected in $G - e$ is the edge $(2, 4)$, which is a cut-edge. The algorithm will then contract the edge $(2, 4)$, then select the edge $(1, 5)$, another cut-edge. After contracting $(1, 5)$ what is left is the triangle on vertexes $1, 3, 4$. For the graph $G \,/\, e$, the MINDEG heuristic selects the edge (2,1). After deleting (2,1), MINDEG will select and contract the edge (2,4) again yielding the triangle $1, 3, 4$. This example shows how identical graphs in the computation tree arise.

2.2 The vertex order heuristic: VORDER

Consider again the graph G shown in the Figure 3.

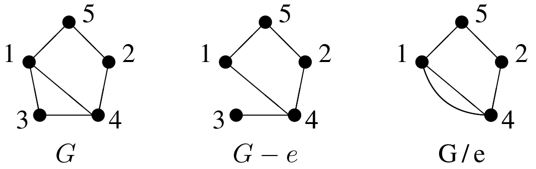


Fig. 3: The VORDER-pull heuristic.

The vertex order heuristic picks the edge $e = (u, v)$ where u is simply the first vertex in the G and v is the first vertex adjacent to u. In our example $u = 1$, $v = 3$, hence $e = (1, 3)$ is chosen. Shown in Figure 3 are the graphs $G - e$ and $G \,/\, e$ where when we contracted the edge $e = (1, 3)$ we "pulled" vertex 3 down to vertex 1. The next edge selected in $G \,/\, e$ will be one of the edges (1,4).

There is alternative choice here when constructing the graph $G \,/\, e$. Instead of "pulling" vertex $v = 3$ down to $u = 1$, if instead we "push" vertex $u = 1$ up to $v = 3$ we get the contracted graph shown in Figure 4. Observe that the two contracted graphs $G \,/\, e$ in figures 3 and 4 are isomorphic. However, in the vertex order heuristic, the next edge selected in $G \,/\, e$ is different. In figure 3 the vertex order heuristic selects edge (1,4). In figure 4 it selects edge (2,4). We will call the two vertex order heuristics VORDER-pull and VORDER-push, respectively.

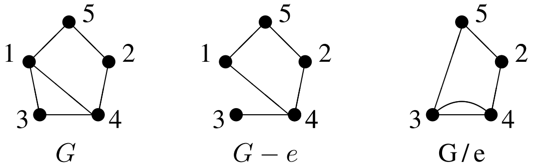


Fig. 4: The VORDER-push heuristic.

To visualize the difference between VORDER-pull and VORDER-push, picture the computation tree of graphs produced by the algorithm as it applies the identity

$$T(G) = T(G - e) + T(G \,/\, e).$$

On the left of the computation tree we repeatedly delete edges. On the right of the tree we repeatedly contract edges. The two vertex order heuristics differ when we contract. In the VORDER-pull heuristic, we always select the same first vertex and contract (pull) other vertexes to it thus typically increasing the degree of the first vertex. In the VORDER-push heuristic, we select the first vertex and push it away (into the middle of the graph) and move on to the next vertex in the ordering. Thus one measurable difference between VORDER-pull and VORDER-push is that the degree of the vertex u selected will generally be greater in VORDER-pull than in VORDER-push. We will measure this explicitly in our benchmarks.

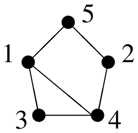
2.3 The vertex label ordering

The VORDER-pull and VORDER-push heuristics, and also to a lesser extent, the MINDEG heuristic, also depend on the input permutation of the labels of the vertexes in G. All three heuristics are sensitive to this ordering with a random ordering producing a bad behavior. In [9], Haggard, Pearce and Royle state "using an ordering where vertexes with higher degree come lower in the ordering generally also gives better performance". Their idea is to increase the probability that more identical graphs appear *higher* in the computation tree. To achieve this we propose to label the vertexes in the input graph in an order so that the algorithm deletes and contracts edges *locally*. We found that the following vertex ordering heuristic works best amongst the orderings we tried. To simplify the presentation we assume G is connected. We describe it below with pseudo-code and an example.

```
Algorithm **SHARC** - short arc order.
Input: An undirected connected graph $G$ on $n > 0$ vertexes $V = \{1, 2, ..., n\}$.

1 Initialize the ordered list $S = [1]$
2 while $|S| < n$ do the following
      Using breadth first search (BFS), starting from the vertexes in $S$ find the first path
      from $S$ back to $S$ which includes at least one new vertex, that is, find a path $u \to v_1 \to v_2 \to ... \to v_m \to w$ where $u \in S$, $w \in S$, $m > 0$, $v_i \in V \backslash S$.
      If such a path exists, append $v_1, v_2, ..., v_m$ to $S$. Otherwise ($G$ may have a cutedge)
      pick the least vertex $v_1$ not in $S$ but adjacent to a vertex in $S$ and append $v_1$ to $S$.
  end while
3 output $S$.
```

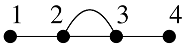
We explain the algorithm with an example. Consider again the graph G below.



Initially we have $S = [1]$. Using BFS we insert all vertexes adjacent to the vertexes in S not already in S into a queue Q. In the example, we obtain $Q = [3, 4, 5]$. Hence we have paths $1 \to 3$, $1 \to 4$ and $1 \to 5$ which we maintain in an array $P = [0, 0, 1, 1, 1]$, that is $P_3 = 1$ stores the edge from 3 to 1 and $P_1 = 0$ indicates the end of a path. We take the first vertex 3 from Q and consider the new edge $(3, 4)$. Since P_4 is not zero we know there is a path from 1 back to 4 stored in P. Since 3 came from Q we know there is a path from 1 to 3 stored in P. Thus we are done this iteration; we extract the path $1 \to 3 \to 4 \to 1$ from P and append $3, 4$ to S obtaining $S = [1, 3, 4]$. In the second iteration the algorithm will find the path $1 \to 5 \to 2 \to 4$ and set $S = [1, 3, 4, 5, 2]$. Since $|S| = 5$ the algorithm stops. The reader can see that the algorithm finds a short cycle in the first iteration, then in the subsequent iterations, finds short arcs from S back to S. We will call this ordering a short arc ordering (SHARC). By picking the first path found using BFS, the short arc ordering maintains locality in S. Although it would be simpler to order the vertexes in simple breadth first search order, that ordering did not prove to be as good as SHARC in our experiments.

3 Maple Implementation

We use a list of neighbors representation for a multi-graph in our Maple implementation. We illustrate with an example in Figure 5.



[[2], [1,3,3], [2,2,4], [3]]

Fig. 5: Maple list of lists data structure for G

To identify identical graphs in the computation tree we make use of `option remember`. This is a feature of the Maple programming language that enables our Maple procedure to automatically identify identical graphs in the computation tree using hashing. For this to work we must canonically re-label vertexes to be $1, 2, ..., n-1$ after edge contraction.

To identify non-equal isomorphic graphs we have implemented our own graph isomorphism test for multi-graphs as the `IsIsomorphic` command in Maple's `GraphTheory` package (see [4]) treats simple graphs only. Instead of searching all previous graphs, we first hash on the characteristic polynomial of the Laplacian matrix of G, a known graph invariant. The Laplacian matrix is an n by n matrix $D-A$ where D is the degree matrix of G and A is the adjacency matrix of G. For increased efficiency, we compute the characteristic polynomial of $D-A$ modulo a machine prime p. This can be computed in $O(n^3)$ arithmetic operations in $\mathbb{F}_p$. See Algorithm 2.2.9 in Chapter 2 of [2].

Our Maple code may be downloaded from `http://www.cecm.sfu.ca/~mmonagan/tutte`

4 Experiments

4.1 Random cubic graphs

In this experiment we generated ten random connected cubic graphs on n vertexes for $16 \leq n \leq 50$. Note, the probability that these graphs are biconnected is high so Theorem 1 is not applicable. Indeed all the graphs generated are biconnected. We computed the average and median time it takes our Maple program to compute the Tutte polynomial using the MINDEG, VORDER-pull and VORDER-push heuristics, on a 2.66 Ghz Intel Core i7 980 desktop with 6 GB RAM. We do this for two permutations of the vertex labels, random (see Table 1) and SHARC (the short arc ordering) (see Table 2). In all cases, we do not use an explicit graph isomorphism test; rather, we use Maple's `option remember;` facility so that Tutte polynomials for identical graphs that appear in the computation tree are not recomputed. The data shows the SHARC ordering is much better than the input random ordering for both VORDER-pull and VORDER-push. The data also shows that VORDER-push with SHARC is much better than VORDER-pull with SHARC. We find similar results for random quartic graphs.

4.2 Generalized Petersen graphs.

The generalized Petersen graph $P(n, k)$ with $1 \leq k < n/2$ is a cubic graph on $2n$ vertexes. Figure 6 shows $P(5, 1)$ and $P(5, 2)$. $P(5, 2)$ is the familiar Petersen graph. To construct $P(n, k)$ the vertexes are divided into two sets $1, 2, ..., n$ and $n+1, n+2, ..., 2n$, which are placed on two concentric circles

		MINDEG heuristic		VORDER pull		VORDER push	
n	m	ave	med	ave	med	ave	med
16	24	0.41	0.36	0.18	0.11	0.22	0.14
18	27	1.21	1.02	0.53	0.33	0.57	0.45
20	30	3.90	3.38	1.27	1.02	1.86	1.46
22	33	14.40	12.07	4.65	3.36	7.22	6.88
24	36	56.24	32.19	13.84	9.23	25.05	22.46
26	39	193.34	118.98	41.03	20.07	58.94	24.57
28	39			199.70	116.32	210.69	75.24

Tab. 1: Timings in CPU seconds for random cubic graphs with n vertices using random vertex order.

		MINDEG heuristic		VORDER pull		VORDER push	
n	m	ave	med	ave	med	ave	med
18	27	0.68	0.51	0.05	0.03	0.02	0.02
22	33	7.73	4.68	0.38	0.14	0.10	0.07
26	39	80.11	38.45	1.24	0.41	0.17	0.12
30	45			11.10	4.36	0.67	0.37
34	51			94.58	19.15	2.06	1.29
38	57					5.40	2.83
42	63					40.66	8.82
46	69					87.63	49.03
50	75					179.64	39.61

Tab. 2: Timings in CPU seconds for random cubic graphs with n vertices using SHARC vertex order.

as shown in figure 6. The first set of vertexes are connected in a cycle $1, 2, ..., n, 1$. The second set are connected to the first with vertex i connected to $n+i$ for $1 \leq i \leq n$. The second parameter governs how the second set is connected. Connect $n+i$ to $n+(n+i \pm k \mod n)$ for $1 \leq i \leq n$.

The SHARC vertex order for $P(5,1)$ and $P(5,2)$ is [1,2,7,6,10,5,4,3,8,9] and [1,5,4,3,2,8,6,9,10,7] respectively. In Tables 3 and 4 we compare the time it takes to compute the Tutte polynomials of $P(n,3)$ for increasing n using the VORDER-pull (Table 3 and VORDER-push (Table 4). For the first set of timings in each table we apply a full graph isomorphism test for $|V| > 15$. For the second set of timings we identify identical graphs only in the computation tree only. Column #calls is the total number of recursive calls made by the algorithm. Column #ident counts the number of recursive calls for which

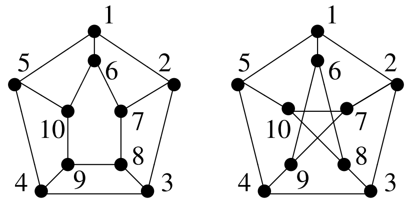


Fig. 6: Petersen graphs $P(5,1)$ and $P(5,2)$.

the graph is identical to a graph previously computed in the computation tree. Column #isom counts the number of recursive calls for which the graph is not identical but isomorphic to a graph previously computed in the computation tree.

VORDER-pull			with isomorphism test				no isomorphism test		
n	$\|V\|$	m	#calls	#ident	#isom	time	#calls	#ident	time
8	16	24	28641	10419	0	1.21	28641	10419	1.19
9	18	27	30235	9818	3	1.40	32693	10681	1.41
10	20	30	90772	31049	22	4.53	240600	85017	12.16
11	22	33	434402	149286	244	26.63	736447	259390	44.82
12	24	36	471530	152284	978	34.72	1217966	406976	87.89
13	26	39	1668636	552034	7072	177.33	5905078	2049833	730.16
14	28	42	4035615	1346519	45340	798.37	17437880	6062683	2805.78
15	30	45	6330229	2016961	149699	2149.02			

Tab. 3: Data for $P(n, 3)$ for VORDER-pull with the SHARC ordering.

VORDER-push			with isomorphism test				no isomorphism test		
n	$\|V\|$	m	#calls	#ident	#isom	time	#calls	#ident	time
8	16	24	2776	703	0	0.10	2776	703	0.09
10	20	30	4680	1119	6	0.31	6490	1634	0.23
12	24	36	7449	1828	16	0.79	9552	2487	0.40
14	28	42	40142	10639	192	7.66	46924	12962	2.34
16	32	48	62306	16316	691	26.04	77896	22103	4.41
18	36	54	88244	23154	1299	55.82	112280	32545	7.30
20	40	60	115682	30503	1996	105.72	148412	43676	11.30
22	44	66	143035	37734	2754	181.40	184852	54925	15.68
24	48	72	170917	45204	3501	289.46	221107	66114	21.70
26	52	78	198675	52641	4278	445.24	257671	77437	29.50
28	56	84	226615	60085	5071	674.25	294126	88700	39.07
30	60	90	254629	67585	5855	975.65	330379	99888	50.28

Tab. 4: Data for $P(n, 3)$ for VORDER-push with the SHARC ordering.

In comparing the data for $P(n, 3)$, it's clear that VORDER-push (Table 4) is much better than VORDER-pull (Table 3). In fact, VORDER-push is polynomial time in n. The reader can see that the number of graphs (column #calls) is increasing linearly with n. We find the same linear increase for VORDER-push for $P(n, 1)$, $P(n, 2)$, $P(n, 3)$ and $P(n, 4)$. For $P(n, 5)$ and $P(n, 6)$ the data is not clear.

The data for $P(n, 3)$ also shows that a high percentage of isomorphic graphs in the computation tree are identical (compare columns #ident and #isom). The data shows that the explicit graph isomorphism test helps VORDER-pull (Table 3) but hurts the performance of VORDER-push (Table 4).

In Table 5 we show data for $P(n, 6)$. The irregularity of the data in Table 5 is partly explained by low girth. In particular, $P(18, 6)$ has 6 triangles. The girth of $P(n, k)$ is a minimum when k divides n where the girth is n/k. In Table 6 we fix n to be 14 and vary k to show the dependence on the girth.

		VORDER-pull			VORDER-push		
k	girth	time(s)	#calls	#ident	time(s)	#calls	#ident
13	5	85.55	875232	270060	0.30	6884	1715
14	6	1262.37	5524084	1807371	4.16	85103	23822
15	5				6.62	124203	35033
16	7				43.69	606569	177341
17	6				23.35	384107	112730
18	3				3.98	65379	16181
19	6				24.55	315584	87375
20	7				482.93	3647975	1081545

Tab. 5: Data for $P(n, 6)$ for VORDER-pull and VORDER-push.

		VORDER-pull			VORDER-push		
k	girth	time(s)	#calls	deg	time(s)	#calls	deg
1	4	6.12	54040	6.48	0.16	693	2.10
2	5	209.33	1362412	5.19	0.65	4727	2.30
3	6	806.92	4035615	4.32	3.82	40142	2.47
4	7	2273.75	8430139	4.61	7.71	88579	2.49
5	6	1218.51	6208087	4.49	5.62	71717	2.50
6	6	979.73	5524084	4.44	6.43	71054	2.47

Tab. 6: Data for $P(14, k)$. Column deg shows the average degree of the first vertex in the computation tree.

4.3 The truncated icosahedron graph.

The Tutte polynomial of a planar graph G and its dual G^* are related by $T(G, x, y) = T(G^*, y, x)$. Shown in Figure 7 is the truncated icosahedron graph TI and its dual TI^*.

In [6], Haggard, Pearce and Royle report that they computed the Tutte polynomial for TI^* in one week on 150 computers. They used the VORDER-pull heuristic. Using the VORDER-push heuristic, and the vertex ordering as shown in the figure 7, we computed the Tutte polynomial for TI on a single core of a 2.66 Ghz Intel Core i7 desktop in under 4 minutes and 2.8 gigabytes, and for TI^* in under 9 minutes and 8.8 gigabytes. Notice that the vertexes of TI (and also TI^*) are numbered in concentric cycles. This was the ordering that we input the graph from a picture. Notice that the vertex ordering is a short arc ordering. This is why we tried the short arc ordering on other graphs.

4.4 Dense graphs.

Up to this point, the data shows that VORDER-push is much better than VORDER-pull. This, however, is not the case for dense graphs. In Table 7 we give data for the complete graphs K_n on n vertexes. VORDER-pull is clearly better than VORDER-push.

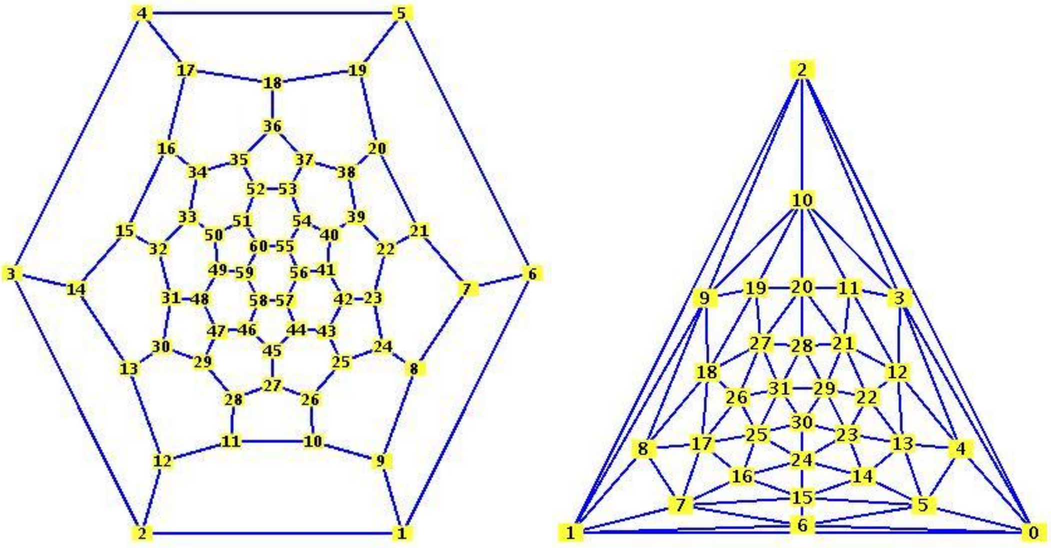


Fig. 7: The truncated icosahedron graph and its dual.

4.5 Monitoring execution for large graphs.

For large graphs, the user of software for computing Tutte polynomials will need some way to know how far a large computation has progressed and how much memory has been consumed so that the user can stop the computation when it becomes obvious that it not going to terminate in a reasonable time. When the Tutte polynomial for a graph G of size n vertexes in the computation tree is computed for the first time, we display the additional time it took to compute $T(G)$ since the time it took to compute the Tutte polynomial for a graph of size $n-1$ for the first time, and the total space used after $T(G)$ is computed. In Table 8 and Table 9 we show the output of VORDER-push (VORDER-pull respectively) for the truncated icosahedron TI (for $n > 20$). The reader can see that VORDER-pull will take a very long time.

		VORDER-pull				VORDER-push			
n	m	time(s)	#calls	#ident	deg	time	#calls	#ident	deg
10	45	0.08	2519	1002	7.40	0.24	7448	2826	4.84
11	55	0.18	5075	2024	8.27	0.64	17178	6667	5.25
12	66	0.46	10191	4070	9.12	1.72	38940	15372	5.70
13	78	1.07	20427	8164	10.02	4.57	87070	34829	6.10
14	91	2.43	40903	16354	10.90	12.64	192544	77838	6.54
15	105	6.10	81859	32736	11.81	39.00	421922	172047	6.95
16	120	16.36	163775	65502	12.71	113.42	917540	376848	7.37
17	136	46.25	327611	131036	13.64	273.40	1982502	819217	7.78
18	153	113.39	655287	262106	14.54				

Tab. 7: Data for K_n for VORDER-pull and VORDER-push

n	time(s)	space	n	time	space	n	time	space	n	time(s)	space
21	0.06	0.067gb	31	0.84	0.100gb	41	4.60	0.258gb	51	25.32	1.167gb
22	0.09	0.090gb	32	0.00	0.100gb	42	5.61	0.340gb	52	22.60	1.443gb
23	0.17	0.095gb	33	1.32	0.120gb	43	0.00	0.340gb	53	0.01	1.443gb
24	0.00	0.095gb	34	0.00	0.120gb	44	1.36	0.360gb	54	0.20	1.451gb
25	0.10	0.095gb	35	0.18	0.120gb	45	5.52	0.443gb	55	25.12	1.786gb
26	0.00	0.095gb	36	0.00	0.120gb	46	10.98	0.619gb	56	12.77	1.950gb
27	0.34	0.095gb	37	1.93	0.150gb	47	0.00	0.619gb	57	0.00	1.950gb
28	0.00	0.095gb	38	0.00	0.150gb	48	12.69	0.757gb	58	6.82	2.058gb
29	0.55	0.095gb	39	3.15	0.193gb	49	10.52	0.880gb	59	43.38	2.679gb
30	0.00	0.095gb	40	0.00	0.193gb	50	0.01	0.880gb	60	8.13	2.761gb

Tab. 8: Trace of time and space for the truncated icosahedron using VORDER-push. Total time 204.58 seconds.

n	time(s)	space	n	time	space	n	time	space	n	time(s)	space
21	0.08s	0.085gb	27	1.04s	0.099gb	33	7.44s	0.241gb	39	205.00s	1.532gb
22	0.00s	0.085gb	28	0.00s	0.099gb	34	0.00s	0.241gb	40	0.10s	1.532gb
23	0.36s	0.095gb	29	1.88s	0.115gb	35	14.77s	0.340gb	41	399.84s	3.115gb
24	0.00s	0.095gb	30	0.00s	0.115gb	36	0.20s	0.341gb	42	0.01s	3.115gb
25	0.73s	0.095gb	31	5.22s	0.160gb	37	0.00s	0.341gb	43	758.37s	6.205gb
26	0.00s	0.095gb	32	0.00s	0.160gb	38	59.27s	0.661gb	44	>1500s	>14gb

Tab. 9: Trace of time (in seconds) and space for the truncated icosahedron using VORDER-pull.

5 Conclusion

We have presented a new edge selection heuristic that we call VORDER-push for computing the Tutte polynomial of a graph using the edge deletion and contraction algorithm. We find that for sparse graphs, VORDER-push outperforms VORDER-pull and the other heuristics considered by Haggard, Pearce and Royle in [9] by several orders of magnitude and which significantly increases the range of graphs that can be computed for what is an NP-hard problem. For some graphs, including grid graphs and the Petersen graphs $P(n, k)$ for $1 \leq k \leq 4$, our new heuristic automatically finds polynomial time constructions for the Tutte polynomial. At this point we only have a partial understanding of why and when VORDER-push is so effective. Graphs with large girth appear to be more difficult.

We are integrating our new heuristic into the `TuttePolynomial` command in Maple's GraphTheory package. This should become available in Maple 17. The overall improvement is huge. For example, for the dodecahedron graph, a cubic graph with 20 vertices and 30 edges, the time to compute the Tutte polynomial improves from 162.093 seconds in Maple 16 to 0.219 seconds. The GraphTheory package [4, 3] has been under development since 2004. We have also installed a command for computing the reliablity polynomial $R_p(G)$ in the package. Our improvement for computing $T(G, x, y)$ will automatically improve Maple's performance for computing the chromatic polynomial $P_\lambda(G)$ and other related polynomials.

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