Calculating cyclotomic polynomials

Andrew Arnold (ada26@sfu.ca)
Michael Monagan (mmonagan@cecm.sfu.ca)

July 17, 2008

Abstract

We present two algorithms to calculate $\Phi_n(z)$, the n_{th} cyclotomic polynomial. The first algorithm calculates $\Phi_n(z)$ by a series of polynomial divisions, optimized using the discrete Fourier transform. The second algorithm calculates $\Phi_n(z)$ as a quotient of sparse power series. These algorithms were used to calculate cyclotomic polynomials of large height and length. In particular we find cyclotomic polynomials $\Phi_n(z)$ of minimal order n whose height is greater than n, n^2, n^3, and n^4, respectively.

1 Introduction

The n_{th} **cyclotomic polynomial**, $\Phi_n(z)$, is the monic polynomial whose $\phi(n)$ distinct roots are exactly the n_{th} primitive roots of unity.

$$\Phi_n(z) = \prod_{\substack{k=1 \\ gcd(k,n)=1}}^{n} \left(z - e^{2\pi i \frac{k}{n}}\right).$$

It is an irreducible polynomial in $\mathbb{Z}$. We write

$$\Phi_n(z) = \sum_{k=1}^{\phi(n)} a_n(k) z^k,$$

For $n < 105$, the coefficients of $\Phi_n(z)$ are strictly $-1, 0$, or 1. Denote by $A(n)$ and $S(n)$ the height and length, respectively, of the n_{th} cyclotomic polynomial. That is,

$$A(n) = \max_{0 \leq k \leq \phi(n)} |a_n(k)|, \qquad S(n) = \sum_{k=0}^{\phi(n)} |a_n(k)|.$$

Paul Erdos [5] proved that $A(n)$ is not bounded above by any polynomial of n. There is a wealth of material on the behaviour of $A(n)$ and the size of cyclotomic polynomial coefficients [3], [2], [10], [11]; however, comparatively little work has gone into actually calculating these values. Koshiba calculated

$A(4849845) = 669606$ [8] and found the coefficients of $\Phi_n(z)$ with degree less than $\phi(n)/10$ for $n = 111546345 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$ [9]. Bosma found the least value of k for which a occurs as $a_n(k)$ for some n, for $|a| < 50$ [4]. We present new results on $A(n)$, $S(n)$, and on the bounds of $a_n(k)$ for fixed k.

1.1 Organization of paper.

We present two algorithms to calculate cyclotomic polynomials. Our first algorithm calculates $\Phi_n(z)$ for squarefree n via a series of polynomial divisions. We use the discrete Fourier transform to do these divisions quickly. The second algorithm calculates $\Phi_n(z)$ as a quotient of sparse power series.

Using these two algorithms, we have produced a wealth of data on the heights and lenghts of cyclotomic polynomials. Amongst our results, we have found:

- $A(n)$ and $S(n)$ for all $n < 5 \cdot 10^6$
- the smallest values of n for which $A(n) > n, n^2$ and n^3 respectively
- an order n for which $A(n) > n^4$
- the smallest n such that $A(n) > 2^{64}$ (machine precision)
- $A(n)$ and $S(n)$ for squarefree $n < 6 \cdot 10^8$ with six or more prime factors
- $A(n)$ and $S(n)$ for squarefree $n < 10^9$ with seven or more prime factors
- the maximum and minimum values of $a_n(k)$ for fixed k, for $k < 138$
- the smallest instance of k for which $a_n(k) = a$ for some n, for $|a| < 248$

1.2 Preliminaries

We are interested in finding cyclotomic polynomials with large coefficients. The following two identities are useful:

Lemma 1. *Let $n > 1$ be odd. Then $\Phi_{2n}(z) = \Phi_n(-z)$.*

Lemma 2. *Let p be a prime that divides n. Then $\Phi_{np}(z) = \Phi_n(z^p)$*

Lemma 1 tells us $A(2n) = A(n)$ and $S(2n) = S(n)$ for odd n. Lemma 2 says that $A(np) = A(n)$ and $S(np) = S(n)$ for p dividing n. For the remainder of the paper, we will be strictly concerned with the calculation of cyclotomic polynomials of squarefree, odd order. Lemmas 1 and 2 provide an easy means of calculating $\Phi_n(z)$ for n even or squarefree, provided we can calculate $\Phi_m(z)$, where m is the product of the odd prime factors of n.

2 Using the discrete Fourier transform to calculate $\Phi_n(z)$

Our first algorithm calculates $\Phi_n(z)$ by a series of polynomial divisions. For primes $p \nmid m$,

$$\Phi_{mp}(z) = \frac{\Phi_m(z^p)}{\Phi_m(z)}. \tag{1}$$

We thus are able to calculate $\Phi_n(z)$ by repeated polynomial division, as detailed in algorithm 1.

Input: $n = p_1 p_2 \cdots p_k$, a product of k distinct primes.
Output: $\Phi_n(z)$, the n_{th} cyclotomic polynomial.
$m \longleftarrow 1$
$\Phi_m(z) \longleftarrow z - 1$
for $j = 1$ **to** k **do**
    $m^* \longleftarrow mp_j$
    $\Phi_{m^*}(z) \longleftarrow \dfrac{\Phi_m(z^{p_j})}{\Phi_m(z)}$
    $m \longleftarrow m^*$

Algorithm 1: solving $\Phi_n(z)$ by repeated division

Our key optimization was to perform the polynomial division in algorithm 1 by way of the discrete Fourier transform (DFT)[6]. Algorithm 2 gives a high-level description of the division calculation. Using the discrete Fourier transform, we can calculate $\frac{\Phi_m(z^p)}{\Phi_m(z)}$ in $\mathcal{O}(N \log(N))$ arithmetic operations, where N is the smallest power of 2 greater than $\phi(m) \cdot p$, the degree of the numerator.

Observe that algorithm 2 requires that B_i is nonzero for ω^i, $0 \le i < N$. This is not a problem for odd orders m, however, so long as our modulus q does not divide the order m. Every power of ω is a $2^k th$ root of unity modulo q for some $k < s$. For $m > 1$ it holds that $\Phi_m(z)$ divides

$$\frac{z^m - 1}{z - 1} = z^{m-1} + z^{m-1} + \cdots + z + 1. \tag{2}$$

Thus if q does not divide m, any root of $\Phi_m(z)$ mod q is necessarily an m_{th} root of unity not equal to 1. Given m is always odd for our purposes, the only m_{th} root of unity that is also a $2^k th$ root of unity modulo q for some k is exactly 1, and so $\Phi_m(\omega^i) \neq 0 \bmod q$ for $0 \le i < N$.

2.1 Implementation Details

Our implementation of the DFT modulo a 42-bit prime q requires $12N$ bytes of storage: $8N$ to store the input polynomial and the result of the DFT, and another $4N$ bytes in which we use for the DFT computation. We require that N is a power of two greater than $\phi(n/p_k)p_k$, the degree of the numerator $\Phi_{n/p_k}(z^{p_k})$.

Input:

- $\Phi_m(z)$
- p, a prime not dividing n
- $N = 2^s$, a power of 2 greater than $\phi(m) \cdot p + 1$, the number of coefficients of $\Phi_m(z^p)$
- q, a prime of the form $q = r \cdot N + 1$
- ω, a primitive N_{th} root of unity modulo q

Output: $\Phi_{mp}(z)$

Calculate A_i and B_i by the DFT:
$A_i \longleftarrow \Phi_n(\omega^{ip}) \bmod q$ for $i = 0, 1, \ldots, N-1$
$B_i \longleftarrow \Phi_n(\omega^{i}) \bmod q$ for $i = 0, 1, \ldots, N-1$
$C_i \longleftarrow \dfrac{A_i}{B_i}$ for $i = 0, 1, \ldots, N-1$ /* $C_i = \Phi_{mp}(\omega^i)$ */
Interpolate C by the inverse discrete fast Fourier transform to get the coefficients of $\Phi_{mp}(z)$.

Algorithm 2: A brief description of polynomial division via the discrete fast Fourier transform

For 32-bit primes q, N is no greater than 2^{27}. For cyclotomic polynomials of larger degree, we require larger primes. Choosing unnecessarily large primes for the DFT, however, would require multiprecision arithmetic. Using 64-bit arithmetic, we are able to do multiplication modulo prime numbers as large as 42-bits (see algorithm 3); however, the multiplication is roughly four times slower than multiplying two integers. We typically use a 32-bit prime unless a 42-bit prime is necessary. Our implementation of algorithm 2 requires $20N$ bytes of storage: $16N$ bytes to store the intermediate results A_i and B_i for $0 \le i < N$, and additional $4N$ bytes of memory for the DFT computation.

Input: $a = a_{41}a_{40} \cdots a_1 a_0, b = b_{41}b_{40} \cdots b_1 b_0$, two 42-bit primes modulo a 42-bit prime q
Output: $c = ab \mod q$
$A_0 \longleftarrow a_{20}a_{19} \cdots a_0$ /* $A_0 = a \mod 2^{21}$ */
$A_1 \longleftarrow a_{41}a_{40} \cdots a_{21}$ /* $A_1 = (a - A_0)/2^{21}$ */
$B_0 \longleftarrow b_{20}b_{19} \cdots b_0$ /* $B_0 = b \mod 2^{21}$ */
$B_1 \longleftarrow b_{41}b_{40} \cdots b_{21}$ /* $B_1 = (b - B_0)/2^{21}$ */
$c \longleftarrow (A_1)(B_1)(2^{21}) + (A_1)(B_0) + (A_0)(B_1)$
$c \longleftarrow c \mod q$
$c \longleftarrow (c)(2^{21}) + (A_0)(B_0) \mod q$

Algorithm 3: multiplication modulo a 42-bit prime

The DFT only gives us the coefficients of $\Phi_n(z)$ modulo a prime q_1. Our resulting polynomial, call it $H_n(z)$, will not equal $\Phi_n(z)$ if $A(n) > \frac{q_1}{2}$. If the height of $H_n(z)$ is close to $\frac{q_1}{2}$, we calculate $\Phi_n(z)$ modulo another prime q_2. We then reconstruct $H_n(z) \equiv \Phi_n(z) \bmod q_1 q_2$ by Chinese remaindering. We do this process with primes $q_1, q_2 \ldots q_l$ until the height of $H_n(z)$ is less than $\frac{q_1 q_2 \cdots q_l}{2}$ by a factor of 2^{20} or more. We then take our solution $H_n(z)$ and use the DFT to test that

$$H_n(\omega^j) \cdot \Phi_{n/p_k}(\omega^j) - \Phi_{n/p_k}(\omega^{j p_k}) \equiv 0 \mod q_{l+1}, \tag{3}$$

for some new prime q_{l+1} with N_{th} primitive root ω. If equation (3) holds for all j, $H_n(z) \equiv \Phi_n(z) \pmod{Q = q_1 q_2 \cdots q_l \cdot q_{l+1}}$. For $q_{l+1} > 2^{40}$, it follows that all of the coefficients of $\Phi_n(z)$, modulo Q, lie in the interval $(\frac{-Q}{2^{60}}, \frac{Q}{2^{60}})$. We consider 60 redundant bits sufficient. All our results obtained by this method have been consistent with results we have obtained by non-modular algorithms. It is very unlikely that $\Phi_n(z)$ would have coefficients strictly in such a small range $\bmod Q$ provided its height $A(n)$ were greater than $\frac{Q}{2}$. Table 1 lists the primes and the primitive roots we used in our computations.

	q	$=$	$r \cdot N + 1$	size of q	ω
q_1	2748779069441	$=$	$5 \cdot 2^{39} + 1$	42 bits	243
q_2	4123168604161	$=$	$15 \cdot 2^{38} + 1$	42 bits	624392905782
q_3	2061584302081	$=$	$15 \cdot 2^{37} + 1$	41 bits	624392905781
q_4	206158430209	$=$	$3 \cdot 2^{36} + 1$	38 bits	10648
q_5	2027224563713	$=$	$59 \cdot 2^{35} + 1$	41 bits	1609812002788

Table 1: primes and the primitive roots used in our DFT calculations

The brunt of the computation takes place in the last division step, as each successive division step involves polynomials of greater degree, which requires us to use larger N for the DFT. For squarefree n with largest prime divisor p, we can compute $\Phi_n(z)$ in $\mathcal{O}(\phi(\frac{n}{p})p \cdot \log(\frac{n}{p}p))$ arithmetic operations. Our implementation of algorithm 2 modulo one 42-bit prime q requires $20N$ bytes of storage: $8N$ bytes are used to store the numerator and the final result; an additional $4N$ bytes are used in the DFT computation.

3 Calculating $\Phi_n(z)$ as a quotient of sparse polynomials

It is well known that

$$\Phi_n(z) = \prod_{d|n} \left(1 - z^d\right)^{\mu(\frac{n}{d})}, \tag{4}$$

where μ is the Mőbius function. For instance, for $n = 105 = 3 \cdot 5 \cdot 7$,

$$\Phi_{105}(z) = \frac{(1 - z^3)(1 - z^5)(1 - z^7)(1 - z^{105})}{(1 - z)(1 - z^{15})(1 - z^{21})(1 - z^{35})}$$

The sparseness of each term in this quotient lends itself to fast polynomial arithmetic. For the purposes of our algorithm, we treat $\Phi_n(z)$ as a power series.

Multiplying a power series $C(z) = \sum_{i=0}^{\infty} c_n z^n$ by $1 - z^d$ is easy:

$$\left(\sum_{i=0}^{\infty} c_n z^n\right)\left(1 - z^d\right) = \sum_{i=0}^{d-1} c_n z^i + \sum_{i=d}^{\infty}(c_i - c_{i-d})z^i \tag{5}$$

To divide by $1 - z^d$ we merely multiply by the power series for $\frac{1}{1-z^d}$:

$$\left(\sum_{i=0}^{\infty} c_n z^n\right)\left(1 + z^d + z^{2d} + \cdots\right) = \sum_{i=0}^{\infty}(c_i + c_{i-d} + c_{i-2d} + \cdots)z^i \tag{6}$$

Observe that the coefficients of $C(z)(1+z^d)$ and $C(n)(1+z^d)^-1$ depend strictly on coefficients of lesser degree in $C(z)$. In addition, we know that the $\phi(n)+1$ coefficients of $\Phi_n(z)$, $\{a_n(0), a_n(1), \ldots a_n(\phi(n))\}$, are palindromic, that is, $a_n(k) = a_n(\phi(n) - k)$. So, to calculate the $\Phi_n(z)$ as a power series, we only need compute the first $\frac{\phi(n)}{2} + 1$ terms. We may select the divisors d of n in any order in algorithm 4. We could select all the values of d correspoding to terms in the denominator first, for instance. That method, however, appears to result in some intermediate terms to become very large, typically larger than $A(n)$. We select d in the order given by algorithm 5, using the bits of iterator i to determine which primes divisors p of n to include in our divisor d.

Input: $n = p_1 p_2 \cdots p_k$, a product of k distinct primes.
Output: $a_n(0), a_n(1), \cdots, a_n(\frac{\phi(n)}{2} + 1)$, the first half of the coefficients of $\Phi_n(z)$

$M \longleftarrow \dfrac{\phi(n)}{2} + 1$
$a_n(0) \longleftarrow 1$
for $1 \le i \le M$ **do**
  $a_n(i) \longleftarrow 0$
for $d|n, d > 0$ **do**
  if $\frac{n}{d}$ *has an even number of prime factors* **then**
    $i \longleftarrow M$
    while $i \ge d$ **do**
      $c_i \longleftarrow a_n(i) - a_n(i-d)$
      $i \longleftarrow i - 1$
  else
    $i \longleftarrow d$
    while $i \le M$ **do**
      $a_n(i) \longleftarrow a_n(i) + a_n(i-d)$
      $i \longleftarrow i + 1$

Algorithm 4: solving $\Phi_n(z)$ as a quotient of sparse power series

Input: $n = p_1 p_2 \cdots p_k$, a product of k distinct primes.
Output: $d_0, d_1, \cdots, d_{2^k-1}$, an ordering of the positive divisors of n

```
for i ⟵ 0 to 2^k − 1 do
    j ⟵ i
    d_i ⟵ 1
    for k ⟵ 1to k do
        if j = 1  mod 2 then
            d_i ⟵ d_i p_k
        j = ⌊j/2⌋
```

Algorithm 5: ordering the divisors of n

3.1 A comparison of the two algorithms.

Calculating $\Phi_n(z)$ for n, a product of k distinct primes takes $\mathcal{O}(2^k\phi(n)$ arithmetic operations and $4\phi(n)$ bytes of memory to store the terms with 64-bit precision. We expect that our second approach is slower for n a product of many distinct small primes; however, we currently cannot calculate $\Phi_n(z)$ for odd n with more than 9 distinct prime factors. Calculating $\Phi_n(z)$ where $n = 3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31$, the product of the smallest ten odd primes, would require $\frac{\phi(n)}{2} = 122624409600$ bytes (approximately 114 GB) of memory merely to store the polynomial coefficients up to 64-bit precision. In practise, the power series method is appreciably faster than the DFT approach.

Our implementation of algorithm 4 has several advantages. First, we can perform the calculations in memory; aside from a small overhead, all the memory used in the power series algorithm is to store the coefficients. The power series algorithm makes better use of the memory used to store terms. Using 64-bits of storage for one term gives us exactly 64-bit precision using algorithm 4, whereas algorithm 1 uses 64-bits to store a 42-bit terms. In addition, the arithmetic operations used in algorithm 4 are strictly additions and substractions, which take fewer CPU cycles than multiplication and division operations. In practise, algorithm 4 is approximately 30 times faster than algorithm 1.

4 Results

4.1 Heights and Lengths of cyclotomic polynomials

To find cyclotomic polynomials with large heights, we strictly looked at cyclotomic polynomials of squarefree, odd order. For odd $n > 1$, it holds that

$$\Phi_{2n}(z) = \Phi_n(-z), \tag{7}$$

and so $A(2n) = A(n)$; moreoever, for any prime p dividing some integer n,

$$\Phi_{np}(z) = \Phi_n(z^p), \tag{8}$$

thus $A(np) = A(n)$. We also considered the number of prime factors of n. Bang showed that for $n = pqr$, a product of three primes with $p < q < r$, that $A(n) < p$. For n, a product of two primes, $A(n) = 1$ [1]. Bloom later proved for $n = pqrs$, a product of four primes with $p < q < r < s$, that $A(n) < p(p-1)(pq-1)$ [3]. Bateman proved a more general albeit slightly weaker result [2]: for $n = p_1 p_2 \cdots p_k$, a product of k distinct primes with $p_1 < p_2 < \cdots p_k$,

$$A(n) \leq \prod_{k=1}^{j-2} p_k^{2^{j-k-1}-1}. \tag{9}$$

For instance, for $n = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5$, $A(n) < p_1^7 \cdot p_2^3 \cdot p_3^1 < n^2$.

Using the two methods detailed in this paper, we have created a library of data on $A(n)$ and $S(n)$. We include here our more noteworthy results. Table 2 shows those cyclotomic polynomials we have found whose height is greater than all those of smaller order. Table 2 also shows the growth of $log_n A(n)$, which we found of interest. Our results include the first instances of n such that $A(n) > n$, $A(n) > n^2$, $A(n) > n^3$, and $A(n) > n^4$. Tables 3 through 8 have large $A(n)$ for n, a product of three distinct primes, up to n, a product of eight primes. Table 9 shows $A(n)$ for n, a product of the s smallest odd primes, for $1 \leq s \leq 9$. Table 10 shows $A(n)$ for various multiples of 43037115. $A(43037115)$ was is first instance of $A(n)$ such that $A(n) > n^2$. We expected that if order n produced a large height, that multiplying n by another prime would result in large heights as well. Table 10 supports this. Indeed, given for primes p such that $p \nmid n$

$$\Phi_n p(z) = \frac{\Phi_n(z^p)}{\Phi_n(z)}, \tag{10}$$

it holds that $A(np) \leq A(n)S(n)$. This follows by inspection of long division. Table 10 notably has the first n we have found for which $A(n) > n^4$.

We include every instance of $A(n)$ which required double-precision (128 bit) arithmetic, for $n < 2 \cdot 10^9$, and every instance of $A(n)$ which required triple-precision (192 bit) arithmetic in tables.

Table 11 presents cyclotomic polynomials of large length. In seems, unsurprisingly, that the cyclotomic polynomials with the greatest heights had the greatest lengths as well.

4.2 Extreme values for the k_{th} cyclotomic polynomial coefficient $a_n(k)$

Tables 12 and 13 give the the maximum and minimum values of the k_{th} cyclotomic polynomial coefficient $a_n(k)$, and the smallest order n for which we obtain those extrema. Algorithm 4 tells us that $a_n(k)$ depends strictly on the divisors of n that are less than or equal to k. To find the extreme values of $a_n(k)$ for fixed k, we calculate $a_n(k)$ for all squarefree n whose prime divisors are not greater than k. It is easy to check by inspection whether a non-squarefree order will produce a larger coefficient. By lemma 2, if p divides n, then $a_{np}(k) = a_n(\frac{k}{p})$

if $p|k$, and 0 otherwise. Thus to prove that some non-squarefree order n will not produce a larger coefficient $a_n(k)$, it suffices to show that for every positive divisor d of k, that

$$\max_n a_n(d) < \max_n a_n(k), \tag{11}$$

and similarly for the minimum values. To find the minimum and maximum values of $a_n(k)$ for $0 \leq k \leq K$, where there are P primes less than or equal to K, we need to calculate the the first $K + 1$ terms of some 2^P cyclotomic polynomials. Tenatively we have results for $K = 138$. We find the $k = 119$ is the smallest $k > 0$ such that $a_n(k) > k$ for some n, as $\min_n a_n(119) = -136$.

Tables 14 and 15 show, for $-248 \leq a \leq 248$, the smallest k such that $a_n(k) = a$, and for that a and k, the smallest order n. We extend results from [4] and [7].

References

[1] A. S. Bang. Om ligningen $\phi_n(x) = 0$. *Nyt Tidsskrift for Mathematik*, (6):6–12, 1895.

[2] P. T. Bateman, C. Pomerance, and R. C. Vaughan. On the size of the coefficients of the cyclotomic polynomial. In *Topics in classical number theory, Vol. I, II (Budapest, 1981)*, volume 34 of *Colloq. Math. Soc. János Bolyai*, pages 171–202. North-Holland, Amsterdam, 1984.

[3] D. M. Bloom. On the coefficients of the cyclotomic polynomials. *Amer. Math. Monthly*, 75:372–377, 1968.

[4] Wieb Bosma. Computation of cyclotomic polynomials with Magma. In *Computational algebra and number theory (Sydney, 1992)*, volume 325 of *Math. Appl.*, pages 213–225. Kluwer Acad. Publ., Dordrecht, 1995.

[5] Paul Erdős and R.C. Vaughn. On the coefficients of the cyclotomic polynomial. *Bull. Amer. Math. Soc.*, 52:179–184, 1946.

[6] Keith O. Geddes, Stephen R. Czapor, and George Labahn. *Algorithms for Computer Algebra*. Kluwer Academic Publishers, 101 Philip Drive, Assinippi Park, Norwell, Massachusetts 02061 USA, 1992.

[7] A. Grytczuk and B. Tropak. A numerical method for the determination of the cyclotomic polynomial coefficients. In *Computational number theory (Debrecen, 1989)*, pages 15–19. de Gruyter, Berlin, 1991.

[8] Yôichi Koshiba. On the calculations of the coefficients of the cyclotomic polynomials. *Rep. Fac. Sci. Kagoshima Univ.*, (31):31–44, 1998.

[9] Yôichi Koshiba. On the calculations of the coefficients of the cyclotomic polynomials. II. *Rep. Fac. Sci. Kagoshima Univ.*, (33):55–59, 2000.

[10] Helmut Maier. The size of the coefficients of cyclotomic polynomials. In *Analytic number theory, Vol. 2 (Allerton Park, IL, 1995)*, volume 139 of *Progr. Math.*, pages 633–639. Birkhäuser Boston, Boston, MA, 1996.

[11] R. Thangadurai. On the coefficients of cyclotomic polynomials. In *Cyclotomic fields and related topics (Pune, 1999)*, pages 311–322. Bhaskaracharya Pratishthana, Pune, 2000.

5 Appendix

Table 2: n such that $A(n) > A(m)$ for $m < n$

n	$A(n)$	$log_n A(n)$
1	1	-
105	2	0.15
385	3	0.18
1365	4	0.19
1785	5	0.21
2805	6	0.23
3135	7	0.24
6545	9	0.25
10465	14	0.29
11305	23	0.34
17255	25	0.33
20615	27	0.33
26565	59	0.40
40755	359	0.55
106743	397	0.52
171717	434	0.50
255255	532	0.50
279565	1182	0.56
327845	31010	0.81
707455	35111	0.77
886445	44125	0.78
983535	59815	0.80
1181895	14102773	1.18
1752465	14703509	1.15
3949491	56938657	1.18
8070699	74989473	1.14
10163195	1376877780831	1.73
13441645	1475674234751	1.71
15069565	1666495909761	1.70
30489585	2201904353336	1.65
37495115	2286541988726	1.63
40324935	2699208408726	1.63
43730115	862550638890874931	2.35
169828113	3148456764091573494 1	2.37
185626077	4233794440280272025 8	2.37
416690995	80103182105128365570406901971	3.35
437017385	86711753206816303264095919005	3.35

Table 3: $A(n)$ for n a product of three distinct odd primes, $n < 5 \cdot 10^6$

n	factorization of n	A(n)
105	$3 \cdot 5 \cdot 7$	2
385	$5 \cdot 7 \cdot 11$	3
2431	$11 \cdot 13 \cdot 17$	4
2717	$11 \cdot 13 \cdot 19$	5
8671	$13 \cdot 23 \cdot 29$	6
17119	$17 \cdot 19 \cdot 53$	7
19499	$17 \cdot 31 \cdot 37$	8
20213	$17 \cdot 29 \cdot 41$	10
34891	$23 \cdot 37 \cdot 41$	12
93439	$41 \cdot 43 \cdot 53$	17
282367	$41 \cdot 71 \cdot 97$	26
617927	$53 \cdot 89 \cdot 131$	28
849647	$73 \cdot 103 \cdot 113$	33
874507	$71 \cdot 109 \cdot 113$	34
1188311	$83 \cdot 103 \cdot 139$	35
1713869	$71 \cdot 101 \cdot 239$	37
1941619	$83 \cdot 149 \cdot 157$	39
2059451	$79 \cdot 131 \cdot 199$	41
2702873	$109 \cdot 137 \cdot 181$	45
2857871	$109 \cdot 157 \cdot 167$	49
3302809	$109 \cdot 157 \cdot 193$	54
4455677	$103 \cdot 181 \cdot 239$	58

Table 4: $A(n)$ for n a product of four distinct odd primes, $n < 5 \cdot 10^6$

n	factorization of n	A(n)
1155	$3 \cdot 5 \cdot 7 \cdot 11$	3
1365	$3 \cdot 5 \cdot 7 \cdot 13$	4
1785	$3 \cdot 5 \cdot 7 \cdot 17$	5
2805	$3 \cdot 5 \cdot 11 \cdot 17$	6
3135	$3 \cdot 5 \cdot 11 \cdot 19$	7
6545	$5 \cdot 7 \cdot 11 \cdot 17$	9
10465	$5 \cdot 7 \cdot 13 \cdot 23$	14
11305	$5 \cdot 7 \cdot 17 \cdot 19$	23
17255	$5 \cdot 7 \cdot 17 \cdot 29$	25
20615	$5 \cdot 7 \cdot 19 \cdot 31$	27
31535	$5 \cdot 7 \cdot 17 \cdot 53$	30
40579	$7 \cdot 11 \cdot 17 \cdot 31$	32
46189	$11 \cdot 13 \cdot 17 \cdot 19$	38
99671	$11 \cdot 13 \cdot 17 \cdot 41$	44
104951	$7 \cdot 11 \cdot 29 \cdot 47$	49
111397	$11 \cdot 13 \cdot 19 \cdot 41$	67
141491	$7 \cdot 17 \cdot 29 \cdot 41$	68
151249	$7 \cdot 17 \cdot 31 \cdot 41$	82
192907	$11 \cdot 13 \cdot 19 \cdot 71$	103
293227	$11 \cdot 19 \cdot 23 \cdot 61$	119
312169	$11 \cdot 13 \cdot 37 \cdot 59$	186
322751	$11 \cdot 13 \cdot 37 \cdot 61$	340
608399	$11 \cdot 19 \cdot 41 \cdot 71$	551
1523591	$17 \cdot 19 \cdot 53 \cdot 89$	1732

Table 5: $A(n)$ for n a product of five distinct odd primes, $n < 10^8$

n	factorization of n	A(n)
15015	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	23
21945	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	25
25935	$3 \cdot 5 \cdot \cdot 7 \cdot 13 \cdot 19$	27
26565	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	59
40755	$3 \cdot 5 \cdot 11 \cdot 13 \cdot 19$	359
106743	$3 \cdot 7 \cdot 13 \cdot 17 \cdot 23$	397
171717	$3 \cdot 7 \cdot 13 \cdot 17 \cdot 37$	434
279565	$5 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	585
327845	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 29$	31010
707455	$5 \cdot 7 \cdot 17 \cdot 29 \cdot 41$	35111
886445	$5 \cdot 7 \cdot 19 \cdot 31 \cdot 43$	44125
2448017	$11 \cdot 13 \cdot 17 \cdot 19 \cdot 53$	63482
6278645	$5 \cdot 19 \cdot 29 \cdot 43 \cdot 53$	66604
7499023	$7 \cdot 17 \cdot 29 \cdot 41 \cdot 53$	70337
7897901	$11 \cdot 19 \cdot 23 \cdot 31 \cdot 53$	755262
10878241	$11 \cdot 19 \cdot 23 \cdot 31 \cdot 73$	799449
11948671	$7 \cdot 17 \cdot 31 \cdot 41 \cdot 79$	1503464
19042309	$11 \cdot 13 \cdot 37 \cdot 59 \cdot 61$	44843069
48063521	$11 \cdot 19 \cdot 41 \cdot 71 \cdot 79$	147933903
61448299	$11 \cdot 19 \cdot 41 \cdot 71 \cdot 101$	243750890

Table 6: $A(n)$ for n a product of six distinct primes, $n < 6 \cdot 10^8$

n	factorization of n	$A(n)$
255255	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	532
285285	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19$	1182
345345	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23$	1311
373065	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19$	5477
636405	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 29$	5907
823515	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31$	9642
982905	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 37$	19204
983535	$3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 29$	59518
1181895	$3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 29$	14102773
1752465	$3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 43$	14703509
3949491	$3 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 37$	56938657
8070699	$3 \cdot 7 \cdot 13 \cdot 17 \cdot 37 \cdot 47$	74989473
10163195	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 31$	1376877780831
13441645	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 41$	1475674234751
15069565	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	1666495909761
37495115	$5 \cdot 7 \cdot 17 \cdot 29 \cdot 41 \cdot 53$	2286541988726
187158797	$7 \cdot 17 \cdot 19 \cdot 23 \cdot 59 \cdot 61$	3699907117060
217873513	$11 \cdot 13 \cdot 17 \cdot 19 \cdot 53 \cdot 89$	18704463463469
420669215	$5 \cdot 19 \cdot 29 \cdot 43 \cdot 53 \cdot 67$	44300418456914
576546773	$11 \cdot 19 \cdot 23 \cdot 31 \cdot 53 \cdot 73$	1114044775319814

Table 7: Large $A(n)$ for n a product of seven distinct primes, $n < 2 \cdot 10^9$

n	factorization of n	$A(n)$
4849845	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	669606
8273265	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 29$	39492584
10555545	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37$	88835350
11565015	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 31$	197756850
12267255	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 43$	310102051
18723705	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 29 \cdot 43$	537611376
23881935	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31$	5090663635
28504245	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 37$	5383094111
30470055	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31 \cdot 37$	9927921683
30489585	$3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 31$	2201904353336
40324935	$3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 41$	2699208408726
43730115	$3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 29 \cdot 37$	862550638890874931
169828113	$3 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 37 \cdot 43$	31484567640915734941
185626077	$3 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 37 \cdot 47$	42337944402802720258
416690995	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 41$	80103182105128365570406901971
437017385	$5 \cdot 7 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 43$	86711753206816303264095919005

Table 8: Large $A(n)$ for n a product of eight distinct primes, $n < 2 \cdot 10^9$

n	factorization of n	$A(n)$
111546435	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	8161018310
140645505	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 29$	117507620687
150345195	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31$	267005512042
179444265	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 37$	6835146783975
256471215	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 29 \cdot 31$	15042686880024
306110805	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 29 \cdot 37$	4722828832054556497
355750395	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 29 \cdot 43$	17166282457296523747
453888435	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 43$	158593117761364835221
723768045	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 43 \cdot 59$	162406736625142681947
849140565	$3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 37 \cdot 43$	184827893605787643634
883631595	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37$	18496650466885634187188 46

Table 9: $A(n)$ for n a product of the smallest odd primes

n	factorization of n	$A(n)$
105	$3 \cdot 5 \cdot 7$	3
1155	$3 \cdot 5 \cdot 7 \cdot 11$	3
15015	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	23
255255	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	532
4849845	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	669606*
111546435	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	8161018310
3234846615	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	2888582082500892851

*(Koshiba,2002).

Table 10: $A(n)$ for n a multiple of $m = 43730115$

n			$A(n)$
$7m$	$=$	306110805	4722828832054556497
$17m$	$=$	743411955	6456302257306534821
$23m$	$=$	1005792645	12657890994364960612 73
$31m$	$=$	1355633565	23402413605839956407 1
$41m$	$=$	1792934715	35986559322549038756
$43m$	$=$	1880394945	6454099703601091156682644618152388897 1563
$47m$	$=$	2055315405	44038002270179294436 9
$53m$	$=$	2317696095	6707596266692301982360203066315311880 3367
$59m$	$=$	2580076785	4417542239616800384985305278811989429 9323
$61m$	$=$	2667537015	5096113463996902098607318665569139393 0064

Table 11: $S(n)$ such that $S(n) > S(m)$ for $n > m$, for $n < 10^9$

n	$S(n)$
8645	10853
8671	11535
10439	14155
10465	20747
11305	48893
17255	81517
20615	98157
26565	152149
33915	167163
36295	187959
40579	190157
40755	1896745
62205	1975949
90915	2343245
106743	6362025
145145	6708527
171717	10258345
215215	10273187
267995	15409275
269997	15904365
279565	27024801
327845	1967355537
350455	2129823789
534905	2220988517
597835	2505846229
707455	4976000157
886445	6564013961
983535	6689950767
1051365	8505111595
1181895	2189485343213
1507935	2282800165309
1752465	3180720401669
2301585	3543866518549
3568565	4300031027027
3949491	32085727023085
4589949	36987514115217
7383831	36999794011177
8070699	89647642675785
10163195	2537194394168988549
13441645	3701817831306521717
15069565	4321747364216785297
21931105	4442144031939491241

Continued on Next Page. . .

Table 11 – Continued

n	$S(n)$
25706905	5399723908418499477
30489585	10666361061590540191
36344245	6078128875662462085
37495115	13915523932865067641
40324935	14360210245699163747
43730115	4324164200335279163572713
50821485	4351970812025001664481353
64841205	4528317974732156715054117
92880645	4806583240553661043777801
98968155	5074045637476887470242065
103395435	7581550465553483759465045
121984005	9053764793361253191029937
169828113	669657915156684858719010851
185626077	929269558387565871528659741
243267297	1146268207629720198825188953
347040057	1170940535267962192355779041
416690995	5501746103867884872877328368176745657
437017385	5897388782588085291629368825547154125
617852165	6407997735029917898749762405654076395
798686945	6766416887316797841172598038974721465
943037515	6778652567504059310781549273387619801

Table 12: The maximum values of $a_n(k)$ for fixed k, $0 \le k \le 138$

k	$\max_n a_n(k)$	n
0	1	1
1	1	2
2	1	3
3	1	5
4	1	5
5	1	7
6	1	7
7	2	210
8	1	11
9	1	11
10	1	11
11	2	770
12	1	13
13	2	858
14	1	17
15	2	1430
16	2	165
17	3	646646
18	3	15015
19	3	646646
20	2	3927
21	3	2124694
22	3	373065
23	4	2124694
24	3	11305
25	3	2124694
26	3	2028117
27	3	2124694
28	4	37182145
29	4	3095547
30	5	37182145
31	4	10214438
32	4	40755
33	4	40755
34	5	275147873
35	5	10015005
36	6	215656441
37	5	13498485
38	5	103488385
39	6	170255085
40	6	17596892049

Continued on Next Page...

Table 12 – Continued

k	$\max_n a_n(k)$	n
41	6	63171570
42	6	955049953
43	7	131104243270
44	6	955049953
45	7	131104243270
46	8	845904650955
47	9	150290230090
48	9	17917712785
49	7	4925015277
50	8	1872852957833
51	8	14849835
52	10	30704573184285
53	13	291583831193655
54	12	25725453208455
55	10	14591862285
56	12	397428761916951765
57	9	18455277027
58	11	728792550255
59	15	442420697228304795
60	13	57927965433905
61	13	127091154560370
62	12	498899935172343705
63	15	378943703223140055
64	12	887324910800312985
65	16	2848916332815
66	15	993969333554296364265
67	15	63625798099535
68	14	3783331790535
69	16	1891748086442047919085
70	24	1091736808985866498455
71	17	25725453208455
72	21	63625798099535
73	21	1189504284417436632645
74	16	111495150862095
75	22	8259588898970029005
76	28	4982833447424642054149905
77	26	1256527270719582196335
78	21	4281025919618354440889355
79	28	1852415459294912176665
80	24	4728312119717787804808605
81	25	1587395172801733690395
82	35	1615936890023221260495
83	34	23806785138997669045785703155

Continued on Next Page...

Table 12 – Continued

k	$\max_n a_n(k)$	n
84	33	22038662790004608958212907905
85	28	1247748737866031606205
86	34	23806785138997669045785703155
87	36	4163737538258947469906085
88	37	6520570107084766792494435
89	49	167385506312292611060919278882805
90	43	20364840299624512075310661735
91	33	1106494163767990292295
92	44	167385506312292611060919278882805
93	48	23154544450258006880147738685
94	49	24828366940638103763049984855
95	55	22659470192539950055627356015
96	53	133532257844637925677812008996395
97	53	33481178119721655445849732005
98	48	143185192146659944401509262658785
99	60	194825753248734022710250308207855
100	70	163957329252276946718326137628485
101	66	162491060750703981848903769983565
102	65	1152783981972759212376551073665878035
102	65	1152783981972759212376551073665878035
103	70	165708705518044654756802854537695
104	65	162491060750703981848903769983565
105	68	177378670868250378885451773144465
106	91	1741311422166780032866743258217820415
107	86	1385931304169497030610010841373583705
108	78	1224090207661795864688502686469952965
109	87	205000226831908928153260689867705
110	86	1385931304169497030610010841373583705
111	86	1594947701085598362329200800551420295
112	109	13999349255222447011223909612254419461265
113	110	13417252766180944472850677946397154449365
114	98	12690998857538106169053450769987651287315
115	104	1200321465765450313917852148868594655
116	108	13417252766180944472850677946397154449365
117	116	1625751858694977007458356707961261115
118	124	1634172597969829945454425214724713205
119	136	12942305765608167677351538904046812698945
120	136	139867498408927468089138080936033904837498615
121	118	143185192146659944401509262658785
122	132	12942305765608167677351538904046812698945
123	153	145000250644117466918097276566714048134287555
124	147	12458136493179608808153387453107143924245
125	162	12942305765608167677351538904046812698945

Continued on Next Page...

Table 12 – Continued

k	$\max_n a_n(k)$	n
126	174	200723846966651809454722059951302256832294262386 5
127	150	1711276734928934819063892274221301582201 5
128	156	171239647584761666539038211239443378819741295
129	187	145000250644117466918097276566714048134287555
130	191	2412369169966182297116384390240421618810142052535
131	196	171239647584761666539038211239443378819741295
132	201	2007238469666518094547220599513022568322942623865
133	194	139867498408927468089138080936033904837498615
134	198	2326975571029326286598990252532796074781464457755
135	213	162938425981534060763635083977029188109663335
136	237	2070458578947353310123511012096109893309491997845
137	248	2433554604816929017282913470206053910267638402385
138	229	153446867186493241690025273259920691714925665

Table 13: The minimum values of $a_n(k)$ for fixed k, $0 \le k \le 138$

k	$\min_n a_n(k)$	n
0	1	1
1	-1	1
2	0	1
3	-1	10
4	-1	15
5	-1	14
6	-1	35
7	-2	105
8	-1	21
9	-1	22
10	-1	33
11	-2	385
12	-1	55
13	-2	429
14	-2	715
15	-2	715
16	-2	1001
17	-3	323323
18	-2	1547
19	-3	323323
20	-3	1062347
21	-3	1062347
22	-3	1062347
23	-4	1062347
24	-3	26565
25	-3	1062347
26	-3	1062347
27	-3	1062347
28	-3	15015
29	-4	2800733
30	-3	2471235
31	-4	5107219
32	-4	6678671
33	-4	81510
34	-4	6678671
35	-5	20030010
36	-3	40755
37	-5	26996970
38	-4	1519035
39	-6	340510170
40	-5	26565

Continued on Next Page...

Table 13 – Continued

k	$\min_n a_n(k)$	n
41	-6	31585785
42	-6	14159145
43	-7	65552121635
44	-7	14849835
45	-7	65552121635
46	-7	13498485
47	-9	75145115045
48	-7	33624726135
49	-7	9850030554
50	-7	18455277027
51	-8	29699670
52	-7	1250072985
53	-13	583167662387310
54	-10	56028687878695
55	-10	29183724570
56	-6	538649335
57	-9	36910554054
58	-9	50252983935
59	-15	884841394456609590
60	-11	63545577280185
61	-13	63545577280185
62	-14	33388354164165
63	-15	757887406446280110
64	-13	27280438185
65	-16	5697832665630
66	-14	63625798099535
67	-15	127251596199070
68	-12	125986461315
69	-16	3783496172884095838170
70	-11	18455277027
71	-17	51450906416910
72	-13	13329164735535
73	-21	2379008568834873265290
74	-14	201714447165
75	-22	165191777979400580 10
76	-18	229741702474515
77	-26	2513054541439164392670
78	-23	63545577280185
79	-28	3704830918589824353330
80	-26	607085786279534126565
81	-25	3174790345603467380790
82	-22	38274454773555
83	-34	4761357027799533809157140 6310

Continued on Next Page. . .

Table 13 – Continued

k	$\min_n a_n(k)$	n
84	-20	1994617781525541315
85	-28	2495497475732063212410
86	-16	141379461795
87	-36	83274750765178949398 12170
88	-26	13934049783798165
89	-49	33477101262458522212183855776 5610
90	-32	447352986437830585
91	-33	2212988327535980584590
92	-28	453425274272685
93	-48	46309088900516013760295477370
94	-30	1324920398170528005
95	-55	45318940385079900111254712030
96	-38	235944728441326905
97	-53	66962356239443310891699464010
98	-46	7094857984230700033335
99	-60	38965150649746804542050061641 5710
100	-37	13554795489066266 7255
101	-66	32498212150140796369780753996 7130
102	-46	4653813117912751575755
103	-70	33141741103608930951360570907 5390
104	-32	88443576399117833985
105	-68	35475734173650075777090354628 8930
106	-31	13123555402670743525495
107	-86	27718626083389940612200216827 47167410
108	-46	15579959491496003211465
109	-87	41000045366381785630652137973 5410
110	-43	3072687731927394735
111	-86	31898954021711967246584016011 02840590
112	-52	13554795489066266 7255
113	-110	26834505532361888945701355892 794308898730
114	-64	2586190168445384205705
115	-104	24006429315309006278357042977 37189310
116	-72	5947619448180595837957305
117	-116	32515037173899540149167134159 22522230
118	-61	169675575906962760966 28095
119	-136	25884611531216335354703077808 093625397890
120	-46	7605359744821280186745
121	-118	28637038429331988880301852531 7570
122	-44	5655497802828347618483 8395
123	-153	29000050128823493383619455313 3428096268575110
124	-79	1577642646972422784180945
125	-162	25884611531216335354703077808 093625397890
126	-87	6678687205516589786366 0005

Continued on Next Page. . .

Table 13 – Continued

k	$\min_n a_n(k)$	n
127	-150	34225534698578696381277845484426031644030
128	-72	201034309740817700757609615
129	-187	290000501288234933836194553133428096268575110
130	-77	20578315170319922124794685
131	-196	342479295169523333078076422478886757639482590
132	-93	10502518274056705259368005
133	-194	279734996817854936178276161872067809674997230
134	-113	45029426842175291089174756155
135	-213	325876851963068121527270167954058376219326670
136	-109	1577642646972422784180945
137	-248	4867109209633858034565826940412107820535276804770
138	-120	66786872055165897863660005

Table 14: Least k for which a occurs as $a_n(k)$, $0 < a \leq 248$

a	k	n
0	1	4
1	0	1
2	7	210
3	17	646646
4	23	2124694
5	30	37182145
6	36	215656441
7	43	131104243270
8	46	845904650955
9	47	150290230090
10	52	30704573184285
11	53	127943760945
12	53	2848916332815
13	53	291583831193655
14	59	159545869898415
15	59	442420697228304795
16	65	2848916332815
17	70	2848916332815
18	70	159545869898415
19	70	176794072049595
20	70	152125131763605
21	70	385941459284265885
22	70	307444891294245705
23	70	961380175077106319535
24	70	1091736808985866498455
25	76	397428761916951765
26	76	961380175077106319535
27	76	1664829083670110943585
28	76	4982833447424642054149905
29	82	397428761916951765
30	82	307444891294245705
31	82	385941459284265885
32	82	993969333554296364265
33	82	346693175289255795
34	82	961380175077106319535
35	82	1615936890023221260495
36	87	4163737538258947469906085
37	88	6520570107084766792494435
38	89	97970304029884898115
39	89	1156915125940246587915
40	89	1352450076803386856295
41	89	961380175077106319535

Continued on Next Page...

Table 14 – Continued

a	k	n
42	89	1189504284417436632645
43	89	1091736808985866498455
44	89	42810259196183544408893555
45	89	39291607755401335279395455
46	89	47021104363021270088456855
47	89	238067851389976690457857031555
48	89	226594701925399500556273560155
49	89	16738550631229261106091927888280555
50	95	39291607755401335279395455
51	95	46328910636965753538391655
52	95	42810259196183544408893555
53	95	203648402996245120753106617355
54	95	240122743831393500589483921955
55	95	226594701925399500556273560155
56	99	45732854928417947620279955
57	99	53462351536037882429341355
58	99	297126358469931405688958835155
59	99	277095368011284344631276217055
60	99	19482575324873402271025030820785555
61	100	45732854928417947620279955
62	100	51517430558119180559854955
63	100	42810259196183544408893555
64	100	39291607755401335279395455
65	100	41637375382589474699060855
66	100	220386627900046089582129079055
67	100	248283669406381037630499848555
68	100	203648402996245120753106617355
69	100	15043507529332627070031985823644555
70	100	16395732925227694671832613762848555
71	106	1189504284417436632645
72	106	58566736088239726171174355
73	106	993969333554296364265
74	106	961380175077106319535
75	106	1091736808985866498455
76	106	49828334474246420541499055
77	106	55442794696696721447583455
78	106	41637375382589474699060855
79	106	46328910636965753538391655
80	106	42810259196183544408893555
81	106	48615040104140635176201155
82	106	47021104363021270088456855
83	106	263741374372186303926154471655
84	106	39291607755401335279395455

Continued on Next Page...

Table 14 – Continued

a	k	n
85	106	22038662790004608958212907905
86	106	20364840299624512075310661735
87	106	22659470192539950055627356015
88	106	133532257844637925677812008996395
89	106	167385506312292611060919278882805
90	106	1200321465765450313917852148868594655
91	106	1741311422166780032866743258217820415
92	112	47021104363021270088845685
93	112	4281025919618354440889355
94	112	27060130261144899606919646415
95	112	28176011921398297528854477195
96	112	24828366940638103763049984855
97	112	22038662790004608958212907905
98	112	23154544450258006880147738685
99	112	22659470192539950055627356015
100	112	23806785138997669045785703155
101	112	20364840299624512075310661735
102	112	156055771216022636033105600875305
103	112	162491060750703981848903769983565
104	112	143185192146659944401509262658785
105	112	150435075293326270700319858236445
106	112	133532257844637925677812008996395
107	112	1200321465765450313917852148868594655
108	112	1271627691454486966229803761672669585
109	112	13999349255222447011223909612254419461265
110	113	13417252766180944472850677946397154449365
111	117	133532257844637925677812008996395
112	117	163957329252276946718326137628485
113	117	150435075293326270700319858236445
114	117	170718456231752284727329277324505
115	117	1473812432648717474051033651142451665
116	117	1625751858694977007458356707961261115
117	118	167385506312292611060919278882805
118	118	156055771216022636033105600875305
119	118	133532257844637925677812008996395
120	118	162491060750703981848903769983565
121	118	1152783981972759212376551073665878035
122	118	143185192146659944401509262658785
123	118	1200321465765450313917852148868594655
124	118	1634172597969829945454425214724713205
125	119	165708705518044654756802854537695
126	119	156055771216022636033105600875305
127	119	180860146700965291740833986868535

Continued on Next Page…

Table 14 – Continued

a	k	n
128	119	150435075293326270700319858236445
129	119	133532257844637925677812008996395
130	119	163957329252276946718326137628485
131	119	1295396433350832517000454299274027895
132	119	1224090207661795864688502686469952965
133	119	1152783981972759212376551073665878035
134	119	1334120788125777515447019781882982445
135	119	12704832265321779279601969382871641823735
136	119	12942305765608167677351538904046812698945
137	123	22659470192539950055627356015
138	123	25527757558684247531023223865
139	123	20364840299624512075310661735
140	123	182431394520139137897855843276765
141	123	165708705518044654756802854537695
142	123	156055771216022636033105600875305
143	123	133532257844637925677812008996395
144	123	143185192146659944401509262658785
145	123	183059549453324739044967538335915
146	123	1224090207661795864688502686469952965
147	123	1152783981972759212376551073665878035
148	123	1271627691454486966229803761672669585
149	123	1200321465765450313917852148868594655
150	123	11992411764462614086353260819346129198105
151	123	13156723586255100890853577403748666013455
152	123	12458136493179608808153387453107143924245
153	123	145000250644117466918097276566714048134287555
154	125	1334120788125777515447019781882982445
155	125	1271627691454486966229803761672669585
156	125	1200321465765450313917852148868594655
157	125	1152783981972759212376551073665878035
158	125	1295396433350832517000454299274027895
159	125	1224090207661795864688502686469952965
160	125	11992411764462614086353260819346129198105
161	125	13476009096148710674355726075347712191685
162	125	12942305765608167677351538904046812698945
163	126	172143995052726000572601023645955
164	126	133532257844637925677812008996395
165	126	1509315110417942473936309137686252685
166	126	1200321465765450313917852148868594655
167	126	1271627691454486966229803761672669585
168	126	1152783981972759212376551073665878035
169	126	1385931304169497030610010841373583705
170	126	11992411764462614086353260819346129198105

Continued on Next Page...

Table 14 – Continued

a	k	n
171	126	12458136493179608808153387453107143924245
172	126	147710535702886017701613113511886273333059285
173	126	139867498408927468089138080936033904837498615
174	126	2007238469666518094547220599513022568322942623865
175	129	1402785327460827475301586246268116645
176	129	1561365646469433363598619808636062655
177	129	133532257844637925677812008996395
178	129	1411836562191356788191506371118884335
179	129	1295396433350832517000454299274027895
180	129	1152783981972759212376551073665878035
181	129	1342933917143523618541755374476744515
182	129	1271627691454486966229803761672669585
183	129	13156723586255100890853577403748666013455
184	129	12458136493179608808153387453107143924245
185	129	13938311126032631636844879031694131321185
186	129	153446867186493241690025273259920691714925665
187	129	145000250644117466918097276566714048134287555
188	130	11992411764462614086353260819346129198105
189	130	168097635702472461648413656904774692969837785
190	130	145000250644117466918097276566714048134287555
191	130	2412369169966182297116384390240421618810142052535
192	131	11992411764462614086353260819346129198105
193	131	13444919581748290693947715172165135522205
194	131	12690998857538106169053450769987651287315
195	131	139867498408927468089138080936033904837498615
196	131	171239647584761666539038211239443378819741295
197	132	13228742874201027909688648532680781692755
198	132	13970541540044076203689881160681573189545
199	132	145000250644117466918097276566714048134287555
200	132	162938425981534060763635083977029188109663335
201	132	2007238469666518094547220599513022568322942623865
202	135	1389136078357768340826278329589497185
203	135	1152783981972759212376551073665878035
204	135	1489555553901703401608900859439340355
205	135	1224090207661795864688502686469952965
206	135	1634172597969829945454425214724713205
207	135	1200321465765450313917852148868594655
208	135	15701405093677855556359423959350086682055
209	135	13156723586255100890853577403748666013455
210	135	17225813355199977455035096188413201893905
211	135	13970541540044076203689881160681573189545
212	135	189845688987658951531941795092501898072727005
213	135	162938425981534060763635083977029188109663335

Continued on Next Page...

Table 14 – Continued

a	k	n
214	136	174099019721489953731830847172515
215	136	180860146700965291740833986868535
216	136	1402785327460827475301586246268116645
217	136	150435075293326270700319858236445
218	136	133532257844637925677812008996395
219	136	1470056626611618923787032407041312555
220	136	1389136078357768340826278329589497185
221	136	1443083110527002062800114381224040765
222	136	1557386723242012127180321460924954885
223	136	1200321465765450313917852148868594655
224	136	1295396433350832517000454299274027895
225	136	1224090207661795864688502686469952965
226	136	1271627691454486966229803761672669585
227	136	1385931304169497030610010841373583705
228	136	13228742874201027909688648532680781692755
229	136	1152783981972759212376551073665878035
230	136	12690998857538106169053450769987651287315
231	136	11992411764462614086353260819346129198105
232	136	12458136493179608808153387453107143924245
233	136	13444919581748290693947715172165135522205
234	136	12704832265321779279601969382871641823735
235	136	139867498408927468089138080936033904837498615
236	136	156485419011968355386857456888830012342943995
237	136	2070458578947353310123511012096109893309491997845
238	137	13417252766180944472850677946397154449365
239	137	1200321465765450313917852148868594655
240	137	14513086842570059745580750331970177973605
241	137	13156723586255100890853577403748666013455
242	137	12458136493179608808153387453107143924245
243	137	13228742874201027909688648532680781692755
244	137	11992411764462614086353260819346129198105
245	137	145000250644117466918097276566714048134287555
246	137	139867498408927468089138080936033904837498615
247	137	2007238469666518094547220599513022568322942623865
248	137	2433554604816929017282913470206053910267638402385

Table 15: Least k for which a occurs as $a_n(k)$, $-248 \leq a < 0$

a	k	n
-1	1	1
-2	7	105
-3	17	323323
-4	23	1062347
-5	35	20030010
-6	39	340510170
-7	43	65552121635
-8	47	23473865415
-9	47	75145115045
-10	53	2500145970
-11	53	255887521890
-12	53	5697832665630
-13	53	583167662387310
-14	59	319091739796830
-15	59	884841394456609590
-16	65	5697832665630
-17	71	51450906416910
-18	73	304250263527210
-19	73	693386350578511590
-20	73	614889782588491410
-21	73	2379008568834873265290
-22	75	16519177797940058010
-23	77	23287042104429810
-24	77	798048441231871830
-25	77	1987938667108592728530
-26	77	2513054541439164392670
-27	79	794857523833903530
-28	79	3704830918589824353330
-29	83	2183473617971732996910
-30	83	1987938667108592728530
-31	83	2513054541439164392670
-32	83	7858321551080267055879090
-33	83	9723008020828127035240230
-34	83	47613570277995338091571406310
-35	87	9146570985683589524055990
-36	87	8327475076517894939812170
-37	89	30158628299179590
-38	89	195940608059769796230
-39	89	2313830251880493175830
-40	89	2704900153606773712590
-41	89	1922760350154212639070
-42	89	2379008568834873265290

Continued on Next Page…

Table 15 – Continued

a	k	n
-43	89	2183473617971732996910
-44	89	8562051839236708881778710
-45	89	7858321551080267055879090
-46	89	9404220872604254017691370
-47	89	47613570277995338091571406310
-48	89	45318940385079900111254712030
-49	89	334771012624585222121838557765610
-50	95	7858321551080267055879090
-51	95	9265782127393150707678330
-52	95	8562051839236708881778710
-53	95	40729680599249024150621323470
-54	95	48024548766278700117896784390
-55	95	45318940385079900111254712030
-56	99	9146570985683589524055990
-57	99	10692470307207576485868270
-58	99	59425271693986281137791767030
-59	99	55419073602256868926255243410
-60	99	389651506497468045420500616415710
-61	101	10177170533366247498597510
-62	101	40729680599249024150621323470
-63	101	52748274874437260785230894330
-64	101	44077325580009217916425815810
-65	101	286370384293319888803018525317570
-66	101	324982121501407963697807539967130
-67	103	7858321551080267055879090
-68	103	44077325580009217916425815810
-69	103	40729680599249024150621323470
-70	103	331417411036089309513605709075390
-71	107	9404220872604254017691370
-72	107	9734935652830778591611410
-73	107	9265782127393150707678330
-74	107	7858321551080267055879090
-75	107	8562051839236708881778710
-76	107	11469918501618987369929970
-77	107	47613570277995338091571406310
-78	107	8327475076517894939812170
-79	107	49656733881276207526099969710
-80	107	46309088900516013760295477370
-81	107	44077325580009217916425815810
-82	107	48024548766278700117896784390
-83	107	40729680599249024150621323470
-84	107	267064515689275851355624017992790
-85	107	312111542432045272066211201750610

Continued on Next Page…

Table 15 – Continued

a	k	n
-86	107	2771862608338994061220021682747167410
-87	109	410000453663817856306521379735410
-88	113	10823725909978481039229690
-89	113	11650004961584374380125130
-90	113	10438665940987220417511030
-91	113	9734935652830778591611410
-92	113	8562051839236708881778710
-93	113	9265782127393150707678330
-94	113	8327475076517894939812170
-95	113	9456624239435575609617210
-96	113	7858321551080267055879090
-97	113	50456171488621925440321938030
-98	113	51055515117368495062046447730
-99	113	47613570277995338091571406310
-100	113	49656733881276207526099969710
-101	113	45318940385079900111254712030
-102	113	40729680599249024150621323470
-103	113	44077325580009217916425815810
-104	113	325599204059528092748637501388470
-105	113	286370384293319888803018525317570
-106	113	267064515689275851355624017992790
-107	113	312111542432045272066211201750610
-108	113	2305567963945518424753102147331756070
-109	113	2668241576251555030894039563765964890
-110	113	26834505532361888945701355892794308898730
-111	117	267064515689275851355624017992790
-112	117	327914658504553893436652275256970
-113	117	300870150586652541400639716472890
-114	117	341436912463504569454658554649010
-115	117	2947624865297434948102067302284903330
-116	117	3251503717389954014916713415922522230
-117	119	44077325580009217916425815810
-118	119	54103605572136763423959668490
-119	119	48024548766278700117896784390
-120	119	58966851016823214068809975770
-121	119	40729680599249024150621323470
-122	119	324982121501407963697807539967130
-123	119	50456171488621925440321938030
-124	119	286370384293319888803018525317570
-125	119	331417411036089309513605709075390
-126	119	312111542432045272066211201750610
-127	119	361720293401930583481667973737070
-128	119	300870150586652541400639716472890

Continued on Next Page...

Table 15 – Continued

a	k	n
-129	119	267064515689275851355624017992790
-130	119	327914658504553893436652275256970
-131	119	2590792866701665034000908598548055790
-132	119	2448180415323591729377005372939905930
-133	119	2305567963945518424753102147331756070
-134	119	2668241576251555030894039563765964890
-135	119	25409664530643558559203938765743283647470
-136	119	25884611531216335354703077808093625397890
-137	123	45318940385079900111254712030
-138	123	51055515117368495062046447730
-139	123	40729680599249024150621323470
-140	123	364862789040278275795711686553530
-141	123	331417411036089309513605709075390
-142	123	312111542432045272066211201750610
-143	123	267064515689275851355624017992790
-144	123	286370384293319888803018525317570
-145	123	366119098906649478089935076671830
-146	123	2448180415323591729377005372939905930
-147	123	2305567963945518424753102147331756070
-148	123	2543255382908973932459607523345339170
-149	123	2400642931530900627835704297737189310
-150	123	23984823528925228172706521638692258396210
-151	123	26313447172510201781707154807497332026910
-152	123	24916272986359217616306774906214287848490
-153	123	290000501288234933836194553133428096268575110
-154	125	2668241576251555030894039563765964890
-155	125	2543255382908973932459607523345339170
-156	125	2400642931530900627835704297737189310
-157	125	2305567963945518424753102147331756070
-158	125	2590792866701665034000908598548055790
-159	125	2448180415323591729377005372939905930
-160	125	23984823528925228172706521638692258396210
-161	125	26952018192297421348711452150695424383370
-162	125	25884611531216335354703077808093625397890
-163	129	45318940385079900111254712030
-164	129	40729680599249024150621323470
-165	129	411047952209933753744535950674230
-166	129	391238412344394777660461928954990
-167	129	351619332613316825492313885516510
-168	129	325599204059528092748637501388470
-169	129	386645642117309814649187011123890
-170	129	354757341736500757770903546288930
-171	129	327914658504553893436652275256970

Continued on Next Page...

Table 15 – Continued

a	k	n
-172	129	300870150586652541400639716472890
-173	129	286370384293319888803018525317570
-174	129	344287990105452001145202047291910
-175	129	2805570654921654950603172492536233290
-176	129	3122731292938866727197239617272125310
-177	129	267064515689275851355624017992790
-178	129	2823673124382713576383012742237768670
-179	129	2590792866701665034000908598548055790
-180	129	2305567963945518424753102147331756070
-181	129	2685867834287047237083510748953489030
-182	129	2543255382908973932459607523345339170
-183	129	26313447172510201781707154807497332026910
-184	129	24916272986359217616306774906214287848490
-185	129	27876622252065263273689758063388262642370
-186	129	306893734372986483380050546519841383429851330
-187	129	290000501288234933836194553133428096268575110
-188	131	2400642931530900627835704297737189310
-189	131	2823673124382713576383012742237768670
-190	131	2305567963945518424753102147331756070
-191	131	24916272986359217616306774906214287848490
-192	131	23984823528925228172706521638692258396210
-193	131	26889839163496581387895430344330271044410
-194	131	25381997715076212338106901539975302574630
-195	131	279734996817854936178276161872067809674997230
-196	131	342479295169523333078076422478886757639482590
-197	135	324982121501407963697807539967130
-198	135	267064515689275851355624017992790
-199	135	3387208519831270748864075926944253410
-200	135	286370384293319888803018525317570
-201	135	3425636542746341345338589278793517330
-202	135	2778272156715536681652556659178994370
-203	135	2305567963945518424753102147331756070
-204	135	2979111107803406803217801718878680710
-205	135	2448180415323591729377005372939905930
-206	135	3268345195939659890908850429449426410
-207	135	2400642931530900627835704297737189310
-208	135	31402810187355711112718847918700173364110
-209	135	26313447172510201781707154807497332026910
-210	135	34451626710399954910070192376826403787810
-211	135	27941083080088152407379762321363146379090
-212	135	379691377975317903063883590185003796145454010
-213	135	325876851963068121527270167954058376219326670
-214	137	40729680599249024150621323470

Continued on Next Page…

Table 15 – Continued

a	k	n
-215	137	3874316213520480660511165331444 70
-216	137	399786364478154746792630856135210
-217	137	391450728476061864315777670208610
-218	137	334771012624585222121838557765610
-219	137	376817056383498803967524299359690
-220	137	361720293401930583481667973737070
-221	137	327914658504553893436652275256970
-222	137	325599204059528092748637501388470
-223	137	344287990105452001145202047291910
-224	137	312111542432045272066211201750610
-225	137	300870150586652541400639716472890
-226	137	366119098906649478089935076671830
-227	137	267064515689275851355624017992790
-228	137	286370384293319888803018525317570
-229	137	2927294156470152606708994861218971190
-230	137	2771862608338994061220021682747167410
-231	137	2943318027411509157790332302298538590
-232	137	2668241576251555030894039563765964890
-233	137	2685867834287047237083510748953489030
-234	137	2543255382908973932459607523345339170
-235	137	2305567963945518424753102147331756070
-236	137	2448180415323591729377005372939905930
-237	137	2979111107803406803217801718878680710
-238	137	26834505532361888945701355892794308898730
-239	137	2400642931530900627835704297737189310
-240	137	29026173685140119491161500663940355947210
-241	137	26313447172510201781707154807497332026910
-242	137	24916272986359217616306774906214287848490
-243	137	26457485748402055819377297065361563385510
-244	137	23984823528925228172706521638692258396210
-245	137	290000501288234933836194553133428096268575110
-246	137	279734996817854936178276161872067809674997230
-247	137	4014476939333036189094441199026045136645885247730
-248	137	4867109209633858034565826940412107820535276804770