ODE trends in computer algebra: Four linear and nonlinear challenges

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Abstract

This paper discusses possible solving strategies regarding four ODE problems, all relevant and currently out of the scope of the Maple ODE solver. The problems are:

1. Transformations linearizing non-linear first order ODEs
2. Non-linear 2nd order ODEs for which point symmetries are of no use
3. Hypergeometric solutions for 2nd order linear ODEs
4. Solutions for third and higher order linear ODEs with coefficients possibly non-rational

1 Introduction

The ODE solver of the Maple 8 system, viewed in some forums as the most advanced in current computer algebra systems (CAS), contains implementations of varied and cutting edge "exact solution" algorithms. Such an ODE library proved to be a valuable tool for modeling in diverse areas. It is clear however that the algorithms available are covering only a small fraction of the problems popping up in applications. Hence, besides foreseeable random progress, it is interesting to discuss which would be interesting directions for developing further work in the area. Possible criteria for new solving algorithms could then be that they:

- cover currently unsolvable ODE families of varied types appearing in real applications;
- can be implemented in the framework of current computer algebra systems;
- run in reasonable time in the framework of current PC hardware.

Bearing this in mind, this paper briefly discuss four different ODE problems for which solving algorithms matching these criteria could be developed:

1. A relatively wide transformation linearizing non-linear first order ODEs;
2. Non-linear 2nd order ODEs for which point symmetries are of no use;
3. Hypergeometric solutions for 2nd order linear ODEs;

4. Solutions for third and higher order linear ODEs with coefficients possibly non-rational.

Regarding linearization of first order ODEs, one typical type of equation appearing in modeling is "Abel" [1]. One reasonable evolving idea is then to start with generalizations of the methods currently known for that problem [2, 3]; this is discussed in sec. 2.

Non-linear second order ODEs appear all around in connection with the fact that nature phenomena is frequently formulated in terms of second order equations. For this type of problem, the symmetry or integrating factor approaches already cover an important portion of the problem [4, 5] but sometimes - each day more frequently - we find ODE families for which the determination of point symmetries is of no use or this type of symmetry simply doesn't exist, leaving us with no more than resorting to numerical approximations; a discussion with some examples is presented in sec. 3.

For second order linear ODEs, there already exist Kovacic's algorithm [6, 7], to compute Liouvillian solutions whenever the ODE coefficients are rational and this type of solution exists. One natural direction to evolve is then in the development of algorithms to compute hypergeometric (non-Liouvillian) solutions [8, 9]; we discuss this problem in sec. 4.

Due to computational limitations, the implementation of algorithms for third and higher order linear ODEs is currently restricted to just a few of them. Although for this problem there exists theory extending Kovacic's approach [10], higher order equations have class properties not admitted by second order ones. It is natural then to consider taking advantage of this to formulate solving strategies, including in that the case of non-rational coefficients - this is discussed in sec. 5.

The state of the art regarding these problems is that current computer algebra systems - not just Maple - have poor performance with all of them. On the other hand, from some point of view, solving algorithms for them seem to be just "one step" out of the scope of the currently implemented algorithms. These considerations make these four problems some of the main computer algebra ODE challenges in the near future.

2 A transformations linearizing some first order ODEs

The problem we want to discuss here is "*To find, when they exist, P and F entering a transformation $\{x \rightarrow P(x)\,y(x),\quad y(x) \rightarrow Q(x)\}$ which linearizes a given nonlinear first order ODE*". The motivation for considering this transformation is in that relevant nonlinear first order families - e.g. Abel equations [3] - fall in this case. Also, a large number of second order nonlinear ODEs not admitting point symmetries can be reduced to this linearizable first order ODE family [11]. A formulation of the problem is as follows. Given an arbitrary first order ODE

$$y' = \Phi\,(x, y) \tag{1}$$

where we want to know whether it can be obtained from a linear first order (hence solvable) ODE

$$y' = f\,y + g \tag{2}$$

where $f(x), g(x)$ are arbitrary functions, by means of the transformation

$$\{x \rightarrow P(x)\,y,\quad y \rightarrow Q(x)\} \tag{3}$$

where P and Q are arbitrary functions. If so, (1) has for implicit exact solution the result of changing variables in the solution of (2):

$$Q = \left(\int^{P y} \frac{g(a)}{e^{\int f(a)\,da}} da + C_1 \right) e^{\int^{P y} f(a)\,da} \tag{4}$$

The underlying ODE family we are discussing is obtained by changing variables (3) in (2):

$$y' = \frac{Q'}{(FQ+G)\,P} - \frac{P'\,y}{P} \tag{5}$$

where $F = f(Py)$ and $G = g(Py)$. This is a wide ODE family which includes as members many Abel type equations (third degree) as well as extensions of it to higher degree, all of them out of the scope of current solving algorithms. Also, this family naturally includes members non-polynomial in y. So the problem we want to discuss is: Given an arbitrary ODE, to solve its possible equivalence to (5) under transformations (3); that is, to find, when they exist, the values of each of the functions $\{f, g, P, F\}$ entering the solution (4).

A possible approach to tackle this problem is based on the observation that the transformation (3) admits a decomposition into the following sequence of three transformations. The first one,

$$\{x \rightarrow y, \quad y \rightarrow x\} \tag{6}$$

switches the independent and dependent variables. The second one,

$$\{x \rightarrow Q(x)\} \tag{7}$$

is an arbitrary change of the independent variable. The third one,

$$\{y \rightarrow P(x)\,y\} \tag{8}$$

is an arbitrary linear change of the dependent variable. Performing orderly the sequence (6), (7) and (8) is equivalent to performing (3). This decomposition is of use for tracking the change in the symmetry pattern of the ODE family under (3), in order to detect "when is that computing the ODE symmetry becomes a difficult problem". The symmetry of the departing solvable linear ODE is:

$$[\xi = 0,\, \eta = \tilde{f}(x)] \tag{9}$$

where $\tilde{f}'/\tilde{f} = f$, and is straightforward to compute. Changing variables using (6) we obtain

$$[\xi = \tilde{f}(y),\, \eta = 0] \tag{10}$$

Symmetries of this form are also easy to compute because, as before, the infinitesimals depend on only one variable or are equal to zero. The second change of variables, (7), transforms (10) into

$$[\xi = \frac{\tilde{f}(y)}{F'(x)},\, \eta = 0] \tag{11}$$

where ξ now separates by product. Such a symmetry, when it exists, can also be systematically computed [12]. Concretely, for any equation (1) admitting symmetries of the form $[\xi = \rho(y)\sigma(x),\, \eta = 0]$, the expression

$$\Upsilon(x,y) = \frac{1}{\Phi^2}\left(\frac{\partial^2}{\partial y \partial x}\ln(\Phi)\right) \tag{12}$$

where Φ is the right-hand-side of (1), separates by product[1], the factor depending on x is $\sigma(x)$, and

$$\rho(y) = \sigma\,\frac{\partial}{\partial x}\left(\frac{1}{\sigma\,\Phi}\right) \tag{13}$$

Hence, a possible strategy for solving ODEs belonging to the family (5), so *linearizable* by means of (3) consists of:

[1] We recall that an expression $\Omega(x,y)$ separates by product if and only if $\partial^2 \ln(\Omega(x,y))/\partial x \partial y = 0$.

1. Given (1), change variables in it by means of (8) to obtain

$$y' = -\frac{P' y}{P} + \frac{\Phi(x, Py)}{P} \tag{14}$$

2. For this ODE, compute Υ using formula (12), and determine a value of P - when it exists - such that Υ is separable, so satisfying

$$\frac{\partial^2 \ln(\Upsilon(x,y))}{\partial x \partial y} = 0 \tag{15}$$

3. With such a P in hands, from Υ compute σ and ρ using (13) and verify whether $[\xi = \sigma(x)\rho(y),\, \eta = 0]$ is a symmetry of (14).

When the last step is successful, a solution for (14) is computable and by changing variables in it using the inverse of (8) we obtain the solution to the given ODE (1).

Example

Consider for instance

$$y' = \frac{\left(1 + kx^3\right)^3 y^3 + \left(x^6 k - x^9 k^2 + 2\,x^3\right) y^2 - \left(k\,(b+1)\,x^3 + b - 2\right) x^4 y + a x^6}{x^4\left(\left(1 + kx^3\right) y + x\right)\left(1 + kx^3\right)} \tag{16}$$

This is an Abel equation, depends on 3 symbolic parameters $\{a, b, k\}$, is currently out of the scope of known algorithms and any direct attempt to match against 2-parameterized Abel ODE classes would put the computer in trouble because of unavoidable swell-expression problems related to computing compositions of multivaried resultants [1].

Following the itemization lines above, we change variables in (16) using (8), in this way introducing P, and compute Υ using (12). Next we use (15) to impose the separability of Υ obtaining an equation for P

$$P' = \frac{-kPx^3 + 2\,P}{x + x^4 k} \tag{17}$$

from where

$$P = \frac{C_1\,x^2}{1 + kx^3} \tag{18}$$

and our auxiliary ODE, obtained by changing variables in (16) using (8), becomes

$$y' = \frac{y^3 + a - by}{x\,(yx + 1)} \tag{19}$$

A symmetry $[\xi, \eta]$ for this ODE such that ξ *separates by product* is straightforwardly computable using Υ and (13):

$$\left[\xi = \exp\left(-\sum \frac{\ln(y - R)}{3\,R^2 - b}\right) x^2,\; \eta = 0\right] \tag{20}$$

where R is a root of $x^3 + a - bx = 0$ and the sum is over all the three roots. Using this symmetry, a solution to (19) is

$$C_1 + \int^y \frac{z\left(z^3+a-bz\right)^{-1}}{\left(e^{\int -\left(z^3+a-bz\right)^{-1}dz}\right)}dz + \frac{x^{-1}}{\left(e^{\int^y -\left(z^3+a-bz\right)^{-1}dz}\right)} = 0 \tag{21}$$

Changing variables back in this solution using $y \to y/P$ we obtain the solution to (16).

3 Non-linear 2nd order ODEs

The standard treatment for second order non-linear equations is to search for point symmetries, and, when they exist, use them to integrate the ODE in different manners [13, 14]. However, only few ODEs admit point symmetries, and even when they do so, these symmetries are sometimes of no use or just inconvenient to integrate the equation. We discuss here a couple of examples showing possible alternative approaches.

3.1 Linearization through raising the order

Consider for instance the third order linear ODE

$$y''' + xy = 0 \tag{22}$$

Changing variables using $y = e^{\int u dx}$ we obtain the second order non-linear ODE

$$u'' + 3\,u u' + u^3 + x = 0 \tag{23}$$

By construction, the solution to (23) is $u = y'/y$ and so it is explicit and rational in two integration constants. The solution to the linear ODE (22) in turn can be expressed in compact form using hypergeometric functions:

$$y = C_1\, {}_0\mathrm{F}_2\left([], [1/2, 3/4], -\frac{1}{64}x^4\right) + C_2\, x^2 {}_0\mathrm{F}_2\left([], [5/4, 3/2], -\frac{1}{64}x^4\right) + C_3\, x\, {}_0\mathrm{F}_2\left([], [3/4, 5/4], -\frac{1}{64}x^4\right) \tag{24}$$

A direct use of a symmetry approach to solve (23), however, hangs the computer because its symmetries involve expressions of 3 pages long and depend on 10 different hypergeometric functions, turning their use rather troublesome[2]. This is not an isolated situation but rather typical of nonlinear ODEs which can be mapped into linear ones of higher order through simple transformations. The ODE

$$u'' = \left(-3\,u + 3\,x^{-1}\right)u' - u^3 + 3\,\frac{u^2}{x} - 6\,\frac{u}{x^2} + 6\,x^{-3} \tag{25}$$

for instance, can be mapped by means of $u = y'/y$ into a linear 3rd order equation having for independent solutions $[x, x^2, x^3]$, so (25) has for solution the rational function:

$$u = \frac{2\,C_1\,x + 3\,C_2\,x^2 + 1}{C_1\,x^2 + C_2\,x^3 + x} \tag{26}$$

Even so, tackling this ODE in Mathematica 4.1 will hang the computer while in Maple 8 the symmetry approach will lead to an implicit solution, where $u(x)$ is the uncomputed root of an uncomputed integral, whose simplified (by hand) version is:

$$x + \int^{\frac{u\,x-1}{x}} \frac{1}{a^2} + \frac{1}{a^2\sqrt{1 - C_1\,a^2}}da + C_2 = 0 \tag{27}$$

[2]In Maple, this symmetry can be seen using `> DEtools[symgen](ODE)`.

This suggests that transformations mapping non-linear ODEs into linear ones of higher order should be considered before attempting the computation of symmetries. Whenever this approach succeeds the reward will be - e.g. with the transformation is as in above - an explicit solution rational in the integration constants (best scenario for applying initial conditions to solve IVP/BVP ODE).

3.2 Dynamical symmetries

Dynamical symmetries are those where the infinitesimals depend on the derivatives of the dependent variable of the problem - say $y(x)$. In connection, one can think of these symmetries as related not to point transformations but to extended transformations where the derivatives y', y'', ... are also viewed as variables.

In some cases the ODE admits point symmetries but also dynamical ones, and while its integration using the former lead to untractable solutions, using the latter it leads to far simpler ones. For example

$$y'' = \frac{{y'}^2\,(-2\,y + \ln(y)\,y + 2\,y'\,x)}{y\,(\ln(y)\,y + 2\,y'\,x)} \tag{28}$$

admits two point symmetries

$$[\xi = 0,\, \eta = \ln(y)\,y] \quad [\xi = x,\, \eta = 0] \tag{29}$$

from where a solution in Maple 8 is computed as

$$\ln(\ln(y)) - \int R(x, C_1)\,dx - C_2 = 0 \tag{30}$$

where $R(x, C_1)$ is any of the roots of the fourth degree polynomial in z

$$\left(-x + 16\,C_1\,x^4\right) z^4 + \left(-32\,C_1\,x^3 - 1\right) z^3 + 24\,C_1\,x^2\,z^2 - 8\,C_1\,x\,z + C_1 = 0 \tag{31}$$

An attempt to compute these roots to then compute the integral leads to an expression of 19 pages long, filled with radicals and where all the integrals remained uncomputed. This type of solution is of few use. On the other hand, this ODE also admits dynamical symmetries. To compute them we formulate the overdetermined PDE symmetry system having for solution infinitesimals of the form[3] $[\xi = 0,\, \eta = g(y, y')]$. Solving this system using Maple's **pdsolve** command we find

$$\eta(y, y') = \frac{C_3\,y\ln(y)\,y' + C_2\,\sqrt{y}{y'}^{3/2} + C_1\,y^2}{y'} \tag{32}$$

Equation (28) then admits two independent dynamical symmetries, obtained by taking $C_3 = 0$ and taking one of C_1, C_2 equal to zero while the other equal to one. Taking for instance $C_1 = 1$, $C_2 = 0$ we have

$$X\_dyn := [\xi = 0,\, \eta = \frac{y^2}{y'}] \tag{33}$$

and using it to integrate (28), e.g. by passing it as an argument to the Maple **dsolve** command as in `> dsolve(ODE, HINT=X_dyn);` we obtain a solution free of any integrals and not 19 pages long but presentable in just one line:

$$\frac{2\,\ln(y)}{x^2} + \frac{C_1\,(\ln(y))^2}{2\,x} - \frac{4\,(1 + C_1\,\ln(y)\,x)^{3/2}}{3\,C_1\,x^3} + \frac{4}{3\,C_1\,x^3} - C_2 = 0 \tag{34}$$

[3] In Maple one can construct this system via `> DEtools[gensys](ODE, [0, g(y,_y1)])`.

A search for dynamical symmetries where the infinitesimals depend on at most two variables and hence the PDE linear system is overdetermined can be used to tackle non-linear equations *not admitting point symmetries* in general. For example,

$$y'' = \frac{y'\,(1-y')\,(y'\,x-y)}{x\left(1+y-y'\,x-2\,y'\,y+2\,{y'}^2 x\right)} \tag{35}$$

This ODE admits no point symmetries, Maple 8 or other CAS are unable to solve it, but dynamical symmetries for it, of the form $[\xi = 0,\ \eta = g(x, y')]$, can systematically be computed as in the previous example, resulting in

$$\eta(x, y') = C_2\, e^{-x\left(y'-1\right)y'} + xy'\,(y'-1)\left(C_1 - Ei\left(1, x\,(y'-1)\,y'\right)C_2\right) \tag{36}$$

where Ei is the exponential integral. Among the two dynamical symmetries implied by this result, the one obtained by taking $C_2 = 0, C_1 = 1$ suffices to integrate (35) using the HINT option of Maple's ode solver.

4 Hypergeometric solutions for second order linear ODEs

For 2nd order LODEs Liouvillian solutions can be computed systematically. Hypergeometric solutions can also be computed systematically in Maple 8 but only for some restricted LODE subfamilies. Hypergeometric solutions are essentially non-Liouvillian and related algorithms directly widen the exact solution capabilities.

To analyze the problem consider two arbitrary second order linear ODEs

$$y'' = Ay \tag{37}$$

$$y'' = By \tag{38}$$

where $\{A(x), B(x)\}$ are arbitrary functions and were we assume we can solve one of these ODEs. We want to know whether these equations can be mapped into each other by a change of variables of the form

$$\{x \to F(x),\quad y \to P(x)\,y\} \tag{39}$$

The motivation is that, if determining F and P were possible, then by choosing $A(x)$ as the invariant of an ODE admitting a computable hypergeometric solution, the same transformation (39) will map this solution into the solution of (38). To formulate the problem we change variables - say in (37) - and equate coefficients arriving at the following system for F, P:

$$F''\,P - 2\,F'\,P' = 0, \tag{40}$$

$$F''\,P' - F'\,P'' + A\,(F)\,P{F'}^3 - BF'\,P = 0 \tag{41}$$

Using (40) to eliminate P from (41) lead to a third order non-linear equation for F

$$2\,F'''\,F' - 3\,{F''}^2 - 4\,A\,(F)\,{F'}^4 + 4\,B{F'}^2 = 0 \tag{42}$$

The possibility of formulating the problem in this way already means that the mapping of (37) into (38) by means of (39) is always possible, regardless of the values of $A(x), B(x)$. This in turn suggests that any attempt to compute F in the general case will imply on solving equations (37) and (38) themselves. In fact, a direct attempt to compute point symmetries for (42) leads to the curious result that the infinitesimals $[\xi, \eta]$ are respectively given by the square of the solutions of (38) and (37). So, unless a restriction is artificially introduced in the problem, there is no indication that this problem could be solved. Regarding possible convenient restrictions, in Maple 8, for instance, F is assumed to be of the form

$$F = \frac{a x^k + b}{c x^k + d} \tag{43}$$

where $\{a, b, c, d, k\}$ are arbitrary constants whose values are determined during the solving process. This approach suffices to compute hypergeometric solutions to basically all Kamke's examples admitting ${}_2\mathrm{F}_1, {}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$ hypergeometric solutions and supersedes previous computer algebra programs for tackling this problem. In a more recent work, M.Bronstein showed how to compute a solution to (42) whenever F is rational in x, provided that both (37) and (38) have a regular singularity at 0 and an irregular one at infinity (i.e. ${}_1\mathrm{F}_1$ type of solutions). These two approaches are the edge of the currently implemented algorithms for this problem but it is reasonable to expect more general approaches popping up in the near future.

5 Higher order linear ODEs

The formulation of an equivalence approach for higher order linear ODEs follow the same steps as the ones presented in the previous section. However, there is an important difference: in the higher order case a solution only exists in some cases. This means for instance, that given two - say third order - equations

$$y''' = A_2(x)\, y' + A_1(x)\, y \tag{44}$$
$$y''' = B_2(x)\, y' + B_1(x)\, y \tag{45}$$

the possibility of mapping them into each other by means of (39) depends on the values of the invariants $A_i(x), B_i(x)$. This fact completely changes the panorama since instead of having a single class under (39), in the higher order case there are infinitely many ones, and that splits the general problem into many smaller ones, all of them tractable. Concretely, changing variables in (44) and equating coefficients with (45) we obtain

$$3\, F''\, P - 3\, F'\, P' = 0 \tag{46}$$
$$F'''\, F'\, P + 6\, F''\, F'\, P' - 3\, {F''}^2 P + A_2(F)\, {F'}^4 P - 3\, {F'}^2 P'' - B_2(x)\, {F'}^2 P = 0 \tag{47}$$
$$F'''\, F'\, P' - 3\, {F''}^2 P' + 3\, F''\, F'\, P'' - {F'}^2 P''' + A_2(F)\, {F'}^4 P' + A_1(F)\, P {F'}^5 - B_1(x)\, {F'}^2 P = 0 \tag{48}$$

Using (46) to express P' as a function of F, P and using this with the other two equations of the system above one can completely eliminate P from the problem obtaining two equations for F:

$$3\, {F''}^2 - 2\, F'''\, F' + A_2(F)\, {F'}^4 - B_2(x)\, {F'}^2 = 0 \tag{49}$$
$$-3\, {F''}^3 + 4\, F'''\, F''\, F' - F''''\, {F'}^2 + A_2(F)\, {F'}^4 F'' + A_1(F)\, {F'}^6 - B_1(x)\, {F'}^3 = 0 \tag{50}$$

This result shows that - unlike the second order case - for third or higher order not all linear ODEs can be transformed between themselves by means of (39). Instead, that mapping will be possible only when the two invariants B_1, B_2 can be expressed in terms of A_1, A_2 and a single function $F(x)$ via (49), (50). In fact, F can be eliminated and the system be expressed as a single equation of the form $\Phi(B_1, B_2) = 0$. The problem then will admit a solution only when this relation between the invariants B_i - the coefficients of the given equation in normal form - holds.

The system of equations (49) and (50) can then be tackled as follows. From (49) we isolate F''' and use it to diminish the differential order of (50) surprisingly arriving at fully solvable first order nonlinear ODE for F:

$$\left(A_1(F) - \frac{1}{2}\frac{d}{dF}A_2(F)\right) F'^3 + \frac{1}{2}{B_2}' - B_1 \tag{51}$$

Consequently, in the higher order case it is possible to formulate algorithmically the solving of an equivalence between a given ODE and any other one we know how to solve in a way which is impossible to be done in the second order case. For example, all the ODEs of the form (45) which can be mapped into

$$y''' = y \tag{52}$$

by means of (39) have coefficients satisfying:

$${B_2}''' = \left(3\,B_1 - \frac{3}{2}{B_2}'\right) B_2 + \frac{24\,B_1 {B_1}'' - 7\left({B_2}''\right)^2 - 28\left({B_1}'\right)^2 + 28{B_1}'{B_2}'' - 12{B_2}'{B_1}''}{6\,(2\,B_{1-{B_2}'})} \tag{53}$$

and if this condition is satisfied then F and P entering (39) are given by

$$F'^3 = B_1 - \frac{{B_2}'}{2}, \quad P = F' \tag{54}$$

Example

The following ODE is out of the scope of current computer algebra systems:

$$y''' = \frac{(6 - 4\,x + x^2)\,y'}{(-1+x)^2} - \frac{\left(1 - 6\,x + 15\,x^2 - 20\,x^3 + 15\,x^4 - 6\,x^5 + x^6 + 4\,(e^x)^3 - (e^x)^3\,x\right) y}{(e^x)^3\,(-1+x)^3} \tag{55}$$

Its coefficients however satisfy (53) from where F and P are given by

$$F = \frac{x}{e^x}, \quad P = -\frac{-1+x}{e^x} \tag{56}$$

So by changing variables in the solution to (52) using (39) with these values of F and P above we obtain the solution to (55):

$$y = -\frac{e^{x\left(e^{-x}+1\right)}C_1}{x-1} - \frac{e^{-1/2\,x\left(e^{-x}-2\right)}\sin\left(1/2\,\sqrt{3}xe^{-x}\right)C_2}{x-1} - \frac{e^{-1/2\,x\left(e^{-x}-2\right)}\cos\left(1/2\,\sqrt{3}xe^{-x}\right)C_3}{x-1} \tag{57}$$

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